Parameterisations

- V can be parameterised in many different ways.
  - $U_1 U_2$ is unitary if U_1 and U_2 are unitary matrices
  - Rotation matrices are unitary matrices involving a phase:

$$R_{12}(\theta,\phi)=\begin{pmatrix} \cos\vartheta & \sin\theta\, e^{i\phi} & 0 \\ -\sin\theta\, e^{-i\phi} & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{23}(\theta,\phi)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\theta\, e^{i\phi} \\ 0 & -\sin\theta\, e^{-i\phi} & \cos\vartheta \end{pmatrix}$$

$$R_{13}(\theta,\phi)=\begin{pmatrix} \cos\vartheta & 0 & \sin\theta\, e^{i\phi} \\ 0 & 1 & 0 \\ -\sin\theta\, e^{-i\phi} & 0 & \cos\vartheta \end{pmatrix}$$

Parameterisations

- All KM parameterisations have the form:

$$V = R_{ij}(\vartheta_1, \phi_1) R_{kl}(\vartheta_2, \phi_2) R_{mn}(\vartheta_3, \phi_3)$$

  - Only one of ϕ_1, ϕ_2 and ϕ_3 can be non-zero as there is only one phase for three families.
  - $R_{kl} = R_{12}, R_{23}$ or R_{31}
  - ij and mn $\neq$ kl

- There are 36 different K-M type parameterisations.

Parameterisations

- The original KM parameterisation is:

$$V_{KM} = R_{23}(\vartheta_{23}, \delta) R_{12}(\vartheta_{12}, 0) R_{13}(\vartheta_{13}, 0)$$

- This parameterisation is unfortunate:
  - Some large M.E. have comparable Re and Im parts.

- The PDG uses the following "standard" parameterisation:
  - the coefficients of the Im parts are all small ($O(10^{-3})$):

$$V_{PDG} = R_{23}(\vartheta_{23}, 0) R_{13}(\vartheta_{13}, -\delta) R_{12}(\vartheta_{12}, 0)$$

Parameterisations

- Expanding:

$$V_{PDG}=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where for the "generation labels" 1,2,3: $c_{ij}=\cos\theta_{ij}, s_{ij}=\sin\theta_{ij}$

- The rotation angles are defined and labelled in a way which relates to the mixing between two generations.
- In the limit $\theta_{23}=\theta_{13}=0$, the situation reduces to the Cabibbo mixing of the first two generations with $\theta_{12}=\theta_c$.

Parameterisations

- Existing measurements of decay processes show:

$$s_{12} \approx 0.22; \quad s_{23} \approx 0.04; \quad s_{13} \approx 0.003$$

hence:

$$s_{12} >> s_{23} >> s_{13}$$

- Wolfenstein used this hierarchy to determine a different parameterisation:
  - $s_{12}=\lambda$

$$V \approx \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)$$

Parameterisations

- A,ρ and η are real numbers which were intended to be of order unity.
- In this approximation, all elements of V are real except V_{ub} and V_{td}, the two smallest elements.
- No physics can depend on the parameterisation used:
  - The invariants of the mixing matrix are:

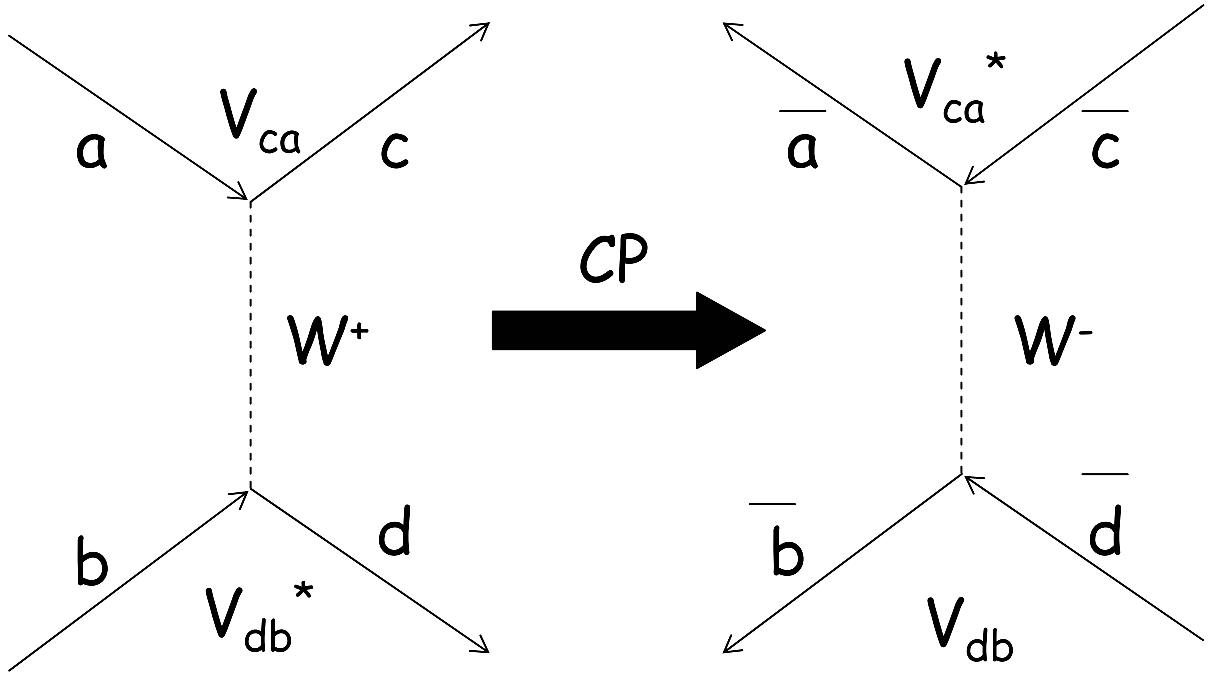
$$\left|V_{ij}\right|$$

$$\mathrm{Im}(V_{\alpha j}V_{\beta k}V_{\alpha k}^{*}V_{\beta j}^{*})\ \mathrm{for}\ \alpha\neq\beta, j\neq k$$

Where is CP violation coming from ?

- CP invariance:

$$A(ab \to cd) = \overline{A}(\overline{a}\overline{b} \to \overline{c}\overline{d})$$



$$A \sim J^{\mu}_{ca}(J^{bd}_{\mu})^{\dagger}$$

$$\sim (\overline{u_c}\gamma^{\mu}(1-\gamma^5)V_{ca}u_a)(\overline{u_b}\gamma_{\mu}(1-\gamma_5)V_{bd}u_d)^{\dagger}$$

$$\sim V_{ca}V^{*}_{db}(\overline{u_c}\gamma^{\mu}(1-\gamma^5)u_a)(\overline{u_d}\gamma_{\mu}(1-\gamma_5)u_b)$$

since: $V^{+}_{bd} = V^{*}_{db}$

Where is CP violation coming from ?

- For the anti-particle process:

$$\overline{A} \sim (J^{\mu}_{ca})^{\dagger}(J^{bd}_{\mu})$$

$$\sim V_{db}V^{*}_{ca}(\overline{u_a}\gamma^{\mu}(1-\gamma^{5})u_c)(\overline{u_b}\gamma_{\mu}(1-\gamma_{5})u_d)$$

Therefore: $\overline{A} = A^{\dagger}$

- To verify if the theory is CP invariant, we need to compute A_{CP} (= CP(A)) and check whether or not $A_{CP} = A^{\dagger}$

Where is CP violation coming from ?

- To compute A_{CP} we need to know how the spinors and the metric transform under C and P:

$$u \xrightarrow{C} u_C = C\overline{u}^T$$

$$\overline{u} \xrightarrow{C} \overline{u}_C = -u^T C^{-1},$$

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T,$$

$$C^{-1}\gamma^\mu\gamma^5 C = (\gamma^\mu\gamma^5)^T$$

Where is CP violation coming from ?

$$(J^{\mu}_{ca})_C = V_{ca}(\overline{u}_c)_C \gamma^{\mu}(1-\gamma^5)(u_a)_C$$

$$= -V_{ca} u_c^T C^{-1} \gamma^{\mu}(1-\gamma^5) C \overline{u}_a^T$$

$$= V_{ca} u_c^T [\gamma^{\mu}(1+\gamma^5)]^T \overline{u}_a^T$$

$$= -V_{ca} \overline{u}_a \gamma^{\mu}(1+\gamma^5) u_c$$

- For the parity operator P:

$$P^{-1}\gamma^{\mu}(1+\gamma^5)P = \gamma^{\mu+}(1-\gamma^5)$$

$$(J^{\mu}_{ca})_{CP} = (-)V_{ca}\overline{u}_a \gamma^{\mu+}(1-\gamma^5)u_c$$

Where is CP violation coming from ?

- Therefore:

$$A_{CP} \sim V_{ca} V_{db}^{*} [\bar{u}_a \gamma^{\mu} (1-\gamma^{5}) u_c][\bar{u}_b \gamma_{\mu} (1-\gamma^{5}) u_d]$$

- Recalling that:

$$\bar{A} \sim V_{db} V_{ca}^{*} (\bar{u}_a \gamma^{\mu} (1-\gamma^{5}) u_c)(\bar{u}_b \gamma_{\mu} (1-\gamma_{5}) u_d)$$

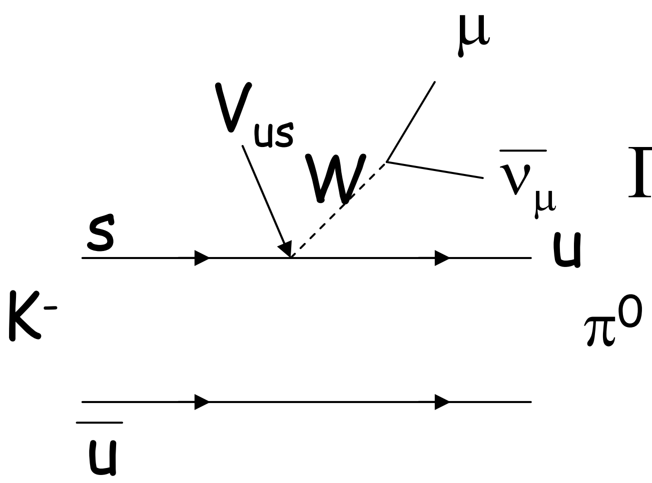
See: f.Halzen, A.D. Martin
"Quarks and Leptons"

$A_{CP} = A^{\dagger} \Leftrightarrow$ the CKM matrix V is real

Since the CKM matrix contains a complex phase factor $e^{i\delta}$, in general we have $A_{CP} \neq A^{\dagger}$ and the theory violates CP invariance.

Determining Elements of V_{CKM}

- $V_{us}=\lambda$
  - Determined from semi-leptonic Kaon decay.



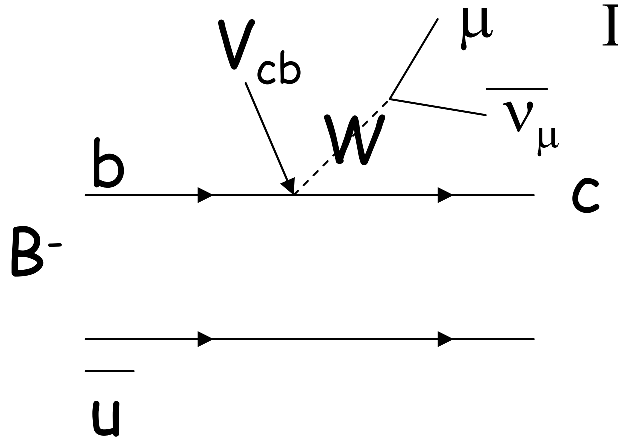
$$\Gamma(K^- \to \pi^0 \mu \bar{\nu}_\mu) \propto |V_{us}|^2$$

- Precision ~ 1%

Determining Elements of V_{CKM}

- $V_{cb}=A\lambda^2$

  - Determined from semi-leptonic B decay.



$$\Gamma(B^- \to X_C l \nu) \propto |V_{cb}|^2$$

$D^0, D^{++}, D^{0++}, \ldots$

- Precision ~ 10%

Determining Elements of V_{CKM}

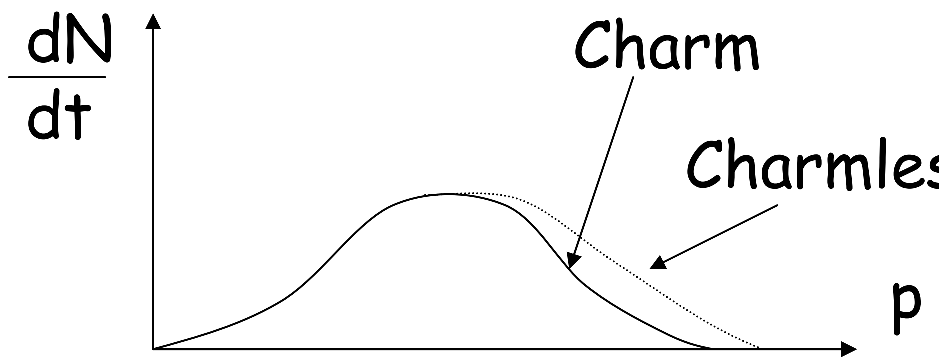
- $V_{ub} = A\lambda^3(\rho - i\eta)$
  - Determined from charmless semi-leptonic B decay.

$$\Gamma(B^- \to X_u l \nu) \propto |V_{ub}|^2$$

- Decays without charm are very rare.
- How do we distinguish them from semi-leptonic decays with charm?

Determining Elements of V_{CKM}

- We use the momentum spectrum of leptons:
  - As $m(X_u) \ll m(X_c)$ (eg $m(\pi^0) \ll m(D^0)$), the leptons from the charmless decay can have a higher momentum.



Determining Elements of V_{CKM}

- Measure shape of p distribution for p>1.4 GeV.
- Fit it to an admixture of the two contributions
  - End point ~2.4GeV/c for charmless decays
  - End point ~3 GeV/c for charm decays
- Then extrapolate to the full range in p
  - This introduces a theoretical error which dominates in the measurement.

$$\frac{|V_{ub}|^2}{|V_{cb}|^2} \text{ determined to } \approx 2\%$$

Determining Elements of V_{CKM}

- All elements of V have been determined to some precision:

(PDG 2000, 90%CL)

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$= \begin{pmatrix} 0.9742-0.9757 & 0.219-0.226 & 0.002-0.005 \\ 0.219-0.225 & 0.9734-0.9749 & 0.037-0.043 \\ 0.004-0.014 & 0.035-0.043 & 0.9990-0.9993 \end{pmatrix}$$

- The further one goes from the leading diagonal, the larger the quark mass and the smaller the mixing angle.

Determining Elements of V_{CKM}

- Two smallest elements (V_{ub} and V_{td}) are most poorly measured.
- In Wolfenstein parameterisation, these elements have the most significant Im part:

$$V_{ub} = A\lambda^3(\rho - i\eta)$$

$$V_{td} = A\lambda^3(1 - \rho - i\eta)$$

Contribute largest to our knowledge of CP violation in S.M.

The Unitarity Triangle

- We have seen that, by definition: $V^{\dagger}V=I$
  - The complex dot product of any pair of rows or columns must equal zero.
  - E.g. first and third columns:

$$V_{ud}V_{ub}^{*}+V_{cd}V_{cb}^{*}+V_{td}V_{tb}^{*}=0$$

$$\lambda_u+\lambda_c+\lambda_t=0$$

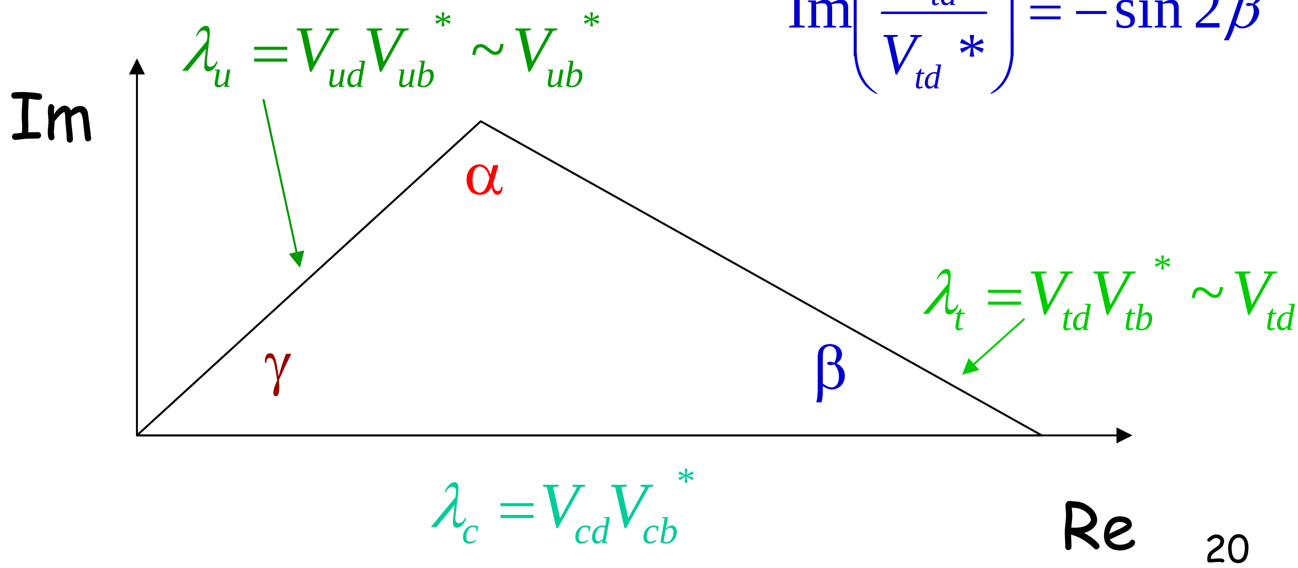
- Each relation represents a triangle in the complex plane.
  - There are six in total.
- The relation shown above is the most interesting for CP violation.

The Unitarity Triangle

- Two of the sides of the triangles are represented by matrix elements with the most significant imaginary parts.
- In the PDG and Wolfenstein parameterisations, λ_c is real.
  - So one side lies along the real axis

$$\mathrm{Im}\left(\frac{V_{ub}{}^*}{V_{ub}}\right) = \sin 2\gamma \qquad \mathrm{Im}\left(\frac{V_{ub}V_{td}}{V_{ub}{}^*V_{td}{}^*}\right) = \sin 2\alpha$$

$$\mathrm{Im}\left(\frac{V_{td}}{V_{td}{}^*}\right) = -\sin 2\beta$$



[Remember that Im(z/z*)=sin(2θ)]

The Unitarity Triangle

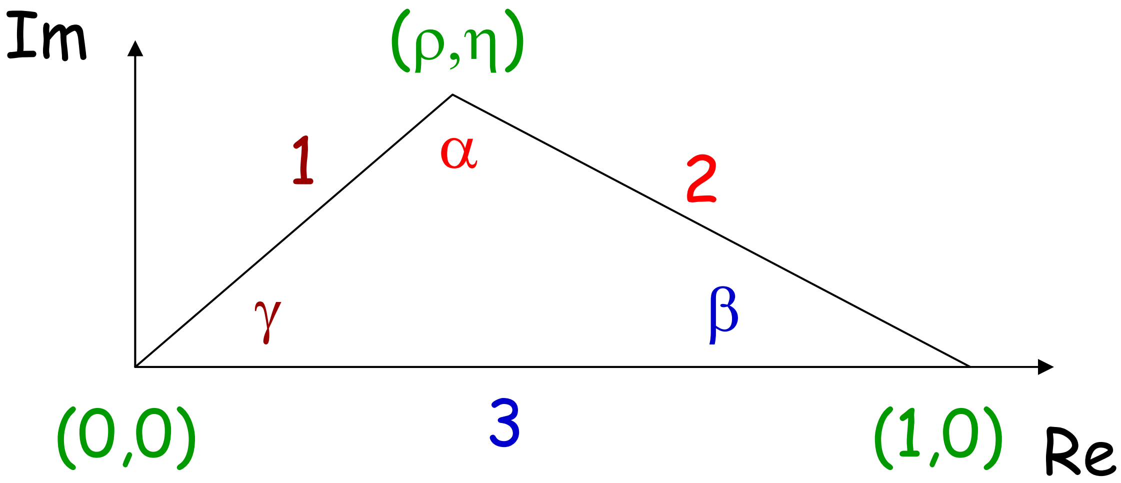
- In the Wolfenstein parameterisation, this unitarity condition reads:

$$A\lambda^3(\rho + i\eta) - A\lambda^3 + A\lambda^3(1 - \rho - i\eta)$$

- (to lowest order in λ) and this is automatically equal to 0.
- If we divide by $A\lambda^3$, the triangle is simply that formed by the three points:

  (0,0); (1,0); (ρ,η)

The Unitarity Triangle



- All six unitarity triangles have different shapes but the same area:

$$\frac{1}{2}A^2\lambda^6\eta$$

  - Side 1 is measured in charmless B decays ($V_{ud}V_{ub}$*)
  - Side 2 is measured in B^0_d mixing.
  - Side 3 is unity by definition
- Studying CP violation with B mesons enables us to measure α,β and γ, sometimes with no theoretical error.