THE STANDARD MODEL

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Contents

1 Introduction, Conventions and the fermionic content of the SM

1.1 Books and papers

- F Halzen and A. D. Martin, Quarks and Leptons, Wiley, 1984.
- M. E. Peskin and D V Schroeder An Introduction to Quantum Field Theory Addison Wesley 1995.
- I J R Aitchison and A J G Hey Gauge theories in particle physics 2nd edition Adam Hilger 1989.
- J D Bjorken and S D Drell Relativistic Quantum Mechanics McGraw-Hill 1964.
- F Mandl and G Shaw Quantum Field Theory Wiley 1984.
- P. Ramond, Journeys beyond the standard model, Perseus Books, 1999.
- D. Balin and A. Love, Introduction to Gauge Field Theory, 1986.
- Several reviews on Spires, e.g. Teubner, Langacker, Signer, Novaes, Evans.
- Also useful S. P. Martin, A Supersymmetry primer

1.2 Overview

- The Standard Model is a description of the strong ($SU(3)_c$), weak ($SU(2)_L$) and electromagnetic ($U(1)_{\rm em}$) interactions in terms of "gauge theories". A gauge theory is one that possesses invariance under a set of "local transformations" i.e. transformations whose parameters are space-time dependent.

- Electromagnetism is a gauge theory, associated with the group $U(1)_{\rm em}$.
In this case the gauge transformations are local complex phase transformations of the fields of charged particles.
Gauge invariance necessitates the introduction of a massless vector (spin-1) particle, the photon, whose exchange mediates the electromagnetic interactions.

- In the 1950's Yang and Mills considered (as a purely mathematical exercise) extending gauge invariance to include local non-Abelian (i.e. non-commuting) transformations such as SU(2).
In this case one needs a set of massless vector fields (three in the case of SU(2)), which were formally called "Yang-Mills" fields, but are now known as "gauge bosons".

- In order to apply such gauge theories to the weak interaction, considers particle transforming into each other under the weak interactions. $\rightarrow$ Doublets of weak isospin.

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

- For $SU(2)_L$ we have three gauge bosons interpreted as the $W^{\pm}-$ and $Z-$ bosons, that mediate weak interactions in the same way that the photon mediates electromagnetic interactions.

- However, weak interactions are known to be short ranged, mediated by very massive vector-bosons, whereas Yang-Mills fields are required to be massless in order to preserve the gauge invariance.

Solved by $\Rightarrow$ the "Higgs mechanism".

- Prescription for breaking the gauge symmetry spontaneously.

- Start with a theory that possesses the required gauge invariance, but the physical quantum states of the theory are *not* invariant under the gauge transformations.

- The breaking of the invariance arises in the quantisation of the theory, whereas the Lagrangian only contains terms which *are* invariant.

- One of the consequences of this is that the gauge bosons acquire a mass and can thus be applied to weak interactions.

- Spontaneous symmetry breaking and the Higgs mechanism has another extremely important consequence. *It leads to a renormalisable theory with massive vector bosons.*

  $\Rightarrow$ Theory is renormalisable

  Infinities that arise in higher order calculations can be re-absorbed into the parameters of the Lagrangian (as in the case of QED).

- If we had simply broken the gauge invariance explicitly by adding mass terms for the gauge bosons, resulting theory would **not** be renormalisable

  $\Rightarrow$ Could not therefore have been used to carry out perturbative calculations.

- Consequence of the Higgs mechanism is the existence of a scalar (spin-0) particle - the Higgs boson.

- Remaining step is to apply the ideas of gauge theories to the strong interactions.
  The gauge theory of strong interactions is called "Quantum ChromoDynamics" (QCD), associated with the group $SU(3)_c$.
  Quarks possess an internal property called "colour" and the gauge transformations are local transformations between quarks of different colours.
  The gauge bosons of QCD are called "gluons" and they mediate the strong interactions.

- The union of QCD and the electroweak gauge theory, which describes the weak and electromagnetic interactions is known as the Standard Model.

- It has eighteen fundamental parameters, most of which are associated with the masses of the gauge bosons, the quarks and leptons, and the Higgs.

- Not all independent and, for example, the ratio of the $W-$ and $Z-$ boson masses are (correctly) predicted by the model.

- Since the theory is renormalisable, perturbative calculations can be performed which predict cross-sections and decay rates both for strongly and weakly interacting processes.
  Predictions have met with considerable success when confronted with ever stringent experimental data.

1.3 Conventions

- Natural units: $\hbar = 1$, $c = 1$

  $\Rightarrow$ energy, momentum, mass, inverse time, inverse length all have same dimensions.

- The metric convention is

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \tag{1.1}$$

- Space time coords and derivatives

$$x^\mu = (x^0, x^i) = (t, \underline{x}); \qquad x_\mu = g_{\mu\nu} x^\nu = (t, -\underline{x}),$$

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \underline{\nabla}\right); \qquad \partial^\mu = \left(\frac{\partial}{\partial t}, -\underline{\nabla}\right).$$

- Lorentz invariant inner product of two vectors:

$$x^2 = g_{\mu\nu} x^\mu x^\nu = x_\mu x^\mu = t^2 - \underline{x}.\underline{x}$$

1.4 Lorentz Invariance

- An arbitrary Lorentz transformation can be written as

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$$

- Lorentz transforms defined by invariance of x^2 under $\Lambda \in$Lorentz Group

$$g_{\mu\nu} x^\mu x^\nu = g_{\rho\sigma} \Lambda^\rho_\mu x^\mu \Lambda^\sigma_\nu x^\nu, \qquad \forall\ x$$

$$\Rightarrow g_{\mu\nu} = g_{\rho\sigma} (\Lambda)^\rho_\mu (\Lambda)^\sigma_\nu.$$

- Klein-Gordon field, $\phi(x)$, transforms as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x)$$

- Transformation should leave KG equation invariant, clearly $\frac{1}{2} m^2 \phi^2(x)$ is invariant, what about the derivatives

$$\partial_\mu \phi(x) \to \partial_\mu \phi(\Lambda^{-1} x) = \frac{\partial}{\partial x^\mu} \phi(\Lambda^{-1} x) = \frac{\partial (\Lambda^{-1})^\nu_\rho x^\rho}{d x^\mu} \frac{\partial}{\partial (\Lambda^{-1})^\nu_\rho x^\rho} \phi(\Lambda^{-1} x)$$

$$= (\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial (\Lambda^{-1})^\nu_\rho x^\rho} \phi(\Lambda^{-1} x) = (\Lambda^{-1})^\nu_\mu (\partial_\nu \phi)(\Lambda^{-1} x).$$

- Transformation of the kinetic term in KG Lagrangian

$$
\begin{aligned}
(\partial_\mu\phi)^2 \rightarrow g^{\mu\nu}(\partial_\mu\phi'(x))(\partial_\nu\phi'(x)) &= g^{\mu\nu}[(\Lambda^{-1})^\rho_\mu(\partial_\rho\phi)][(\Lambda^{-1})^\sigma_\mu(\partial_\sigma\phi)](\Lambda^{-1}x) \\
&= g^{\rho\sigma}(\partial_\rho\phi)(\partial_\sigma\phi)(\Lambda^{-1}x) = (\partial_\rho\phi)^2(\Lambda^{-1}x)
\end{aligned}
$$

- Thus the whole Lagrangian transforms as a scalar

$$\mathcal{L} \rightarrow \mathcal{L}(\Lambda^{-1}x)$$

and with the action, S, being an integral over all space of the Lagrangian it is clear that the action is invariant.

Now Dirac Fermions

- Need a representation for the Lorentz group for Dirac spinors
- General 4-d Lorentz transformation

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

- The Lorentz algebra is given by

$$[L^{\mu\nu}, L^{\rho\sigma}] = i(g^{\nu\rho}L^{\mu\sigma} - g^{\mu\rho}L^{\nu\sigma} - g^{\nu\sigma}L^{\mu\rho} + g^{\mu\sigma}L^{\nu\rho})$$

- We can write down a 4-d matrix representation of the Lorentz generators (as we have Dirac Spinors in mind) defined as

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

where the γ matrices are written in the Weyl or Chiral representation as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \text{where} \qquad \sigma^\mu = (1, \underline{\sigma}) \qquad \text{and} \qquad \bar{\sigma}^\mu = (1, -\underline{\sigma})$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k.$$

- We can then write the spinor form of the Lorentz transformation as:

$$\Lambda_{\frac{1}{2}} = exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right).$$

such that a Dirac fermion, ψ transforms as

$$\psi(x) \to \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x).$$

In addition it can be shown that the γ matrices transform as

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^{\mu}\Lambda_{\frac{1}{2}} = \Lambda^{\mu}_{\nu}\gamma^{\nu} \tag{1.2}$$

- Using this we can show that the Dirac equation is invariant under Lorentz transformations:

$$\begin{aligned}
[i\gamma^{\mu}\partial_{\mu} - m]\,\psi &\to \left[i\gamma^{\mu}(\Lambda^{-1})^{\nu}_{\mu}\partial_{\nu} - m\right]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}\left[i\gamma^{\mu}(\Lambda^{-1})^{\nu}_{\mu}\partial_{\nu} - m\right]\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\left[i\Lambda_{\frac{1}{2}}^{-1}\gamma^{\mu}\Lambda_{\frac{1}{2}}(\Lambda^{-1})^{\nu}_{\mu}\partial_{\nu} - m\right]\psi(\Lambda^{-1}x) \quad \text{using 1.2} \\
&= \Lambda_{\frac{1}{2}}\left[i\Lambda^{\mu}_{\sigma}\gamma^{\sigma}(\Lambda^{-1})^{\nu}_{\mu}\partial_{\nu} - m\right]\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\left[i\gamma^{\mu}\partial_{\mu} - m\right]\psi(\Lambda^{-1}x) = 0
\end{aligned}$$

- Now we need to find how to write the Lagrangian for the Dirac theory, this will involve Dirac bilinears.

- One guess $\psi^{\dagger}\psi$ however this transforms as

$$\psi^{\dagger}\psi \to \psi^{\dagger}\Lambda_{\frac{1}{2}}^{\dagger}\Lambda_{\frac{1}{2}}\psi$$

but $\Lambda_{\frac{1}{2}}^{\dagger} \neq \Lambda_{\frac{1}{2}}^{-1}$ and so $\psi^{\dagger}\psi$ is not Lorentz invariant.

- Better guess is $\bar{\psi} \equiv \psi^{\dagger}\gamma^{0}$ under an infinitesimal Lorentz transformation

$$\bar{\psi} \to \psi^{\dagger}(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^{\dagger})\gamma^{0}$$

- For $\mu \neq 0$ and $\nu \neq 0$, associated with rotations, $(S^{\mu\nu})^{\dagger} = S^{\mu\nu}$ and commutes with γ^{0}

- For $\mu = 0$ or $\nu = 0$, associated with boosts, $(S^{\mu\nu})^{\dagger} = -S^{\mu\nu}$ and anti-commutes with γ^{0}

- this means that

$$\bar{\psi} \to \psi^{\dagger}(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu}))\gamma^{0}$$

$$\therefore \qquad \bar{\psi} \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}$$

- Consequently $\bar{\psi}\psi$ is a Lorentz invariant and the Dirac Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

Weyl and Majoran Spinors

- Much of the standard model (and MSSM) is written in terms of Weyl spinors. Weyl spinners are two-component complex anticommuting objects. We can write a Dirac spinner in terms of two Weyl spinners ξ_α and $(\chi^\dagger)^{\dot{\alpha}} \equiv \chi^{\dagger\dot{\alpha}}$ with two distinct indices $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$:

$$\psi_D = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}$$

  and

$$\bar{\psi}_D = \psi_D^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \chi^\alpha & \xi^\dagger_{\dot{\alpha}} \end{pmatrix}$$

- Undotted indices are used for the first two components of a Dirac spinor
- Dotted indices are used for the last two components of a Dirac spinor
- ξ is called a "left-handed Weyl spinor" and $\chi^\dagger$ is a "right-handed Weyl Spinor"
- We can show this by using:

$$P_L = \frac{1}{2}(1 - \gamma_5) \qquad \text{and} \qquad P_R = \frac{1}{2}(1 + \gamma_5) \qquad \text{with} \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

- We have:

$$P_L \Psi_D = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \qquad P_R \Psi_D = \begin{pmatrix} 0 \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}$$

- Hermitian conjugate of any left-handed Weyl spinner is a right-handed Weyl Spinner:

$$\psi^\dagger_{\dot{\alpha}} \equiv (\psi_\alpha)^\dagger = (\psi^\dagger)_{\dot{\alpha}}$$

  and vice versa

$$(\psi^{\dagger\dot{\alpha}})^\dagger = \psi^\alpha$$

- Raise and lower indices using the antisymmetric symbol

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1, \qquad \epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0$$

  with

$$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \qquad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \qquad \chi^\dagger_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\chi^{\dagger\dot{\beta}}, \qquad \chi^{\dagger\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\chi^\dagger_{\dot{\beta}}$$

- with

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma_\alpha, \qquad \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \epsilon^{\dot\gamma\dot\beta}\epsilon_{\dot\beta\dot\alpha} = \delta^{\dot\gamma}_{\dot\alpha}$$

- The Dirac Lagrangian can then be written as:

$$\begin{aligned}\mathcal{L} = \bar\Psi_D(i\gamma^\mu\partial - m)\Psi_D &= \begin{pmatrix} \chi^\alpha & \xi^\dagger_{\dot\alpha}\end{pmatrix}\left[\begin{pmatrix} 0 & i\sigma^\mu\partial_\mu \\ i\bar\sigma^\mu\partial_\mu & 0\end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix}\right]\begin{pmatrix}\xi_\alpha \\ \chi^{\dagger\dot\alpha}\end{pmatrix} \\ &= i\chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\chi^{\dagger\dot\alpha} + i\xi^\dagger_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu\xi_\alpha - m(\xi\chi + \xi^\dagger\chi^\dagger) \\ &= i\chi^\dagger\bar\sigma^\mu\partial_\mu\chi + i\xi^\dagger\bar\sigma^\mu\partial_\mu\xi - m(\xi\chi + \xi^\dagger\chi^\dagger)\end{aligned}$$

  where the last step involves integration by parts and the identity $\chi\sigma^\mu\chi^\dagger = -\chi^\dagger\bar\sigma^\mu\chi$ etc.

- A four component Majorana spinor can be obtained from the Dirac spinor by imposing the condition $\chi = \xi$ such that,

$$\Psi_D = \begin{pmatrix}\xi_\alpha \\ \xi^{\dagger\dot\alpha}\end{pmatrix}, \qquad \bar\Psi_D = \begin{pmatrix}\xi^\alpha & \xi^\dagger_{\dot\alpha}\end{pmatrix}$$

- The Lagrangian for the Majorana spinor

$$\begin{aligned}\mathcal{L}_M &= \frac{i}{2}\bar\Psi_M\gamma^\mu\partial_\mu\Psi_M - \frac{1}{2}M\bar\Psi_M\Psi_M \\ &= -i\xi^\dagger\bar\sigma^\mu\partial_\mu\xi - \frac{1}{2}M\left(\xi\xi + \xi^\dagger\xi^\dagger\right)\end{aligned}$$

- To efficiently move between the Weyl and Dirac notation we can use the chiral projection operators, $P_{L,R}$ e.g.

$$\bar\Psi_i P_L\Psi_j = \chi_i\xi_j \qquad \text{and} \qquad \bar\Psi_i P_R\Psi_j = \xi^\dagger_i\chi^\dagger_j$$

  and

$$\bar\Psi_i\gamma^\mu P_L\Psi_j = \xi^\dagger_i\bar\sigma^\mu\xi_j \qquad \text{and} \qquad \bar\psi_i\gamma^\mu P_R\Psi_j = \chi_i\sigma^\mu\chi^\dagger_j = -\chi^\dagger_j\bar\sigma^\mu\chi_i$$

1.5 C, P and CP

- How do these fields transform under C, P and CP
- Charge conjugation and Parity are defined by

$$\begin{aligned}C: &\qquad \psi(t,\underline{x}) \to \psi^C(t,\underline{x}) = C\bar\psi^T(t,\underline{x}) \\ P: &\qquad \psi(t,\underline{x}) \to \psi^P(t,\underline{x}) = P\psi(t,-\underline{x})\end{aligned}$$

  with

$$C = i\gamma_2\gamma_0 \qquad \text{and} \qquad P = \gamma_0$$

- Both C and P change the chirality: Consider purely LH state

$$P_L\psi(t,\underline{x}) = \begin{pmatrix} \xi(t,\underline{x}) \\ 0 \end{pmatrix}$$

now act the parity operator on it

$$(P_L\psi(t,\underline{x}))^P = \gamma_0 \begin{pmatrix} \xi(t,-\underline{x}) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \xi(t,-\underline{x}) \end{pmatrix}$$

we can see the final object is purely RH.

- Now for C:

$$(P_L\psi(t,\underline{x}))^C = i\gamma_2\gamma_0 \left[\begin{pmatrix} \xi(t,\underline{x}) \\ 0 \end{pmatrix}^\dagger \gamma_0 \right]^T = i\gamma_2\gamma_0 \begin{pmatrix} 0 \\ \xi^\dagger(t,\underline{x}) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & i\sigma^2 \\ i\bar{\sigma}^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \xi^\dagger(t,\underline{x}) \end{pmatrix} = \begin{pmatrix} 0 \\ i\bar{\sigma}^2\xi^\dagger(t,\underline{x}) \end{pmatrix}$$

the final result being RH.

- Charge conjugation and parity relate the LH and RH components. We will see later that LH and RH states transform differently under the Weak force hence neither C nor P are good symmetries when describing the weak force

- Combined application of both C and P leave a LH (RH) state LH (RH). Acting parity operator on the above we have

$$(P_L\psi(t,\underline{x}))^{CP} = \left[(P_L\psi(t,\underline{x}))^C\right]^P = \begin{pmatrix} 0 \\ i\bar{\sigma}^2\xi^\dagger(t,\underline{x}) \end{pmatrix}^P = \gamma_0 \begin{pmatrix} 0 \\ i\bar{\sigma}^2\xi^\dagger(t,-\underline{x}) \end{pmatrix} = \begin{pmatrix} i\bar{\sigma}^2\xi^\dagger(t,-\underline{x}) \\ 0 \end{pmatrix}$$

which is still LH.

- CP is a better symmetry to use, but see later for CP-violation

- Majorana condition:

$$\psi^C = \psi$$

we can see this by taking the complex conjugate explicitly

$$\psi_M^C = i\gamma_2\gamma_0\bar{\psi}_M^T = i\gamma_2\gamma_0 \begin{pmatrix} \xi^\alpha \\ \xi^\dagger_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} -i\sigma_2\xi^\alpha \\ -i\sigma_2\xi^\dagger_{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \xi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{pmatrix} = \psi_M$$

1.6 Fermionic Content of SM

The fermionic content (in terms of Weyl spinors) of the SM is as follows:

$$
\begin{aligned}
Q_i &= \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}, \\
\bar{u}_i &= \bar{u},\ \bar{c},\ \bar{t}, \\
\bar{d}_i &= \bar{d},\ \bar{s},\ \bar{b}, \\
L_i &= \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}, \\
\bar{e}_i &= \bar{e},\ \bar{\mu},\ \bar{\tau}.
\end{aligned}
$$

"bars" are just labels not conjugates. Left handed states are written in doublets as they transform as doublets under the SM gauge group $SU(2)$, while RH states transform as singlets...more later.

- To form Dirac spinors:

$$
\begin{pmatrix} e \\ \bar{e}^\dagger \end{pmatrix}
$$

- Perhaps more used to writing things in terms of e_L and e_R, with identification $e = e_L$ and $\bar{e} = e_R^\dagger$ such that

$$
\begin{pmatrix} e \\ \bar{e}^\dagger \end{pmatrix} = \begin{pmatrix} (e_L)_\alpha \\ (e_R)^{\dot{\alpha}} \end{pmatrix}
$$

- A more transparent notation is thus:

$$
\begin{aligned}
Q_{Li} &= \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \\
u_{Ri} &= u_R,\ c_R,\ t_R, \\
d_{Ri} &= d_R,\ s_R,\ b_R, \\
L_{Li} &= \begin{pmatrix} \nu_{Le} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{L\mu} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{L\tau} \\ \tau_L \end{pmatrix}, \\
e_{Ri} &= e_R,\ \mu_R,\ \tau_R.
\end{aligned}
$$

Example

- Dirac mass term for electrons

$$
\frac{m_D}{2}\left((e_R^\dagger)^\alpha (e_L)_\alpha + (e_L^\dagger)_{\dot{\alpha}} (e_R)^{\dot{\alpha}}\right)
$$

using

$$e = \begin{pmatrix} (e_L)_\alpha \\ (e_R)^{\dot\alpha} \end{pmatrix} \qquad \bar{e} = \begin{pmatrix} (e_R^\dagger)^\alpha & (e_L^\dagger)_{\dot\alpha} \end{pmatrix}$$

we can write

$$\frac{m_D}{2}\left(\bar{e}P_L e + \bar{e}P_R e\right) = \frac{m_D}{2}\bar{e}e$$

2 Abelian Gauge Symmetry

2.1 Gauge Transformations

- Consider the Lagrangian density for a free Dirac field ψ:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

- This Lagrangian density is invariant under the phase transformation of the fermion field and its conjugate

$$\psi \to e^{i\omega}\psi \qquad \bar{\psi} \to e^{-i\omega}\bar{\psi}$$

- set of phases belong to the group $U(1)$ and is abelian as

$$e^{i\omega_1}e^{i\omega_2} = e^{i\omega_2}e^{i\omega_1}$$

- Infinitesimal form of this transformation

$$e^{i\omega} \approx 1 + i\omega + \mathcal{O}(\omega^2)$$

under which the wavefunction changes by

$$\delta\psi = i\omega\psi$$

and the conjugate

$$\delta\bar{\psi} = -i\omega\bar{\psi}$$

under this lagrangian is invariant up to order ω^2

- Now make the parameter ω depend on space-time

$$\delta\psi(x) = i\omega(x)\psi(x) \qquad \text{and} \qquad \delta\bar{\psi}(x) = -i\omega(x)\bar{\psi}(x)$$

- Such local (space-time) dependent transformations are called "gauge transformations".
- The Lagrangian is now no longer invariant due to the derivative term giving the change in the lagrangian of

$$\begin{aligned}\overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \to \overline{\psi'}\left(i\gamma^\mu\partial_\mu - m\right)\psi' &= \left(\overline{\psi} - i\omega\left(x\right)\right)\left(i\gamma^\mu\partial_\mu - m\right)\left(\psi + i\omega\left(x\right)\psi\right) \\ &= \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + \overline{\psi}\left(\gamma^\mu\partial_\mu\omega(x)\right)\psi\end{aligned}$$

$$\Rightarrow \delta\mathcal{L} = -\bar{\psi}(x)\gamma^\mu(\partial_\mu\omega(x))\psi(x)$$

- However, if we modify the Lagrangian by replacing $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$

$$\overline{\psi}\left(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m\right)\psi = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - eA_\mu\overline{\psi}\gamma^\mu\psi$$

  and <u>demand</u> that A^μ transforms according to:

$$A_\mu \rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\omega\left(x\right)$$

  when

$$\psi \rightarrow \psi' = \left(\psi + i\omega\left(x\right)\psi\right),$$

  then we will have <u>restored</u> local invariance!

- In other words, this is an invariance which only exists if the particles are *not* free! There is an interaction term.

- Interpretation is: e is the electric charge of the fermion field and A_μ is the photon field.

- $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative.

- It is easy to find the way the object $D_\mu\psi$:

$$\left(D_\mu\psi\right)' = D'_\mu\psi' = \left(\partial_\mu + ieA'_\mu\right)e^{i\omega(x)}\psi = e^{i\omega(x)}\partial_\mu\psi + i\left(\partial_\mu\omega\left(x\right)\right)e^{i\omega(x)} + ie\left(A_\mu - \frac{1}{e}\partial_\mu\omega\left(x\right)\right)e^{i\omega(x)}\psi$$

$$\Rightarrow D'_\mu\psi' = e^{i\omega(x)}\left(\partial_\mu + ieA_\mu\right)\psi = e^{i\omega(x)}D_\mu\psi$$
$$\Rightarrow D'_\mu\psi' = e^{i\omega(x)}D_\mu\psi,$$

  so $D_\mu\psi$ transforms like ψ. This means that $\overline{\psi}D_\mu\psi$ is trivially invariant!

- To have a proper QFT we need Kinetic terms for the photon fields, must be gauge invariant. Define field strength,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Easy to see this is invariant but must be Lorentz invariant so we add to the Lagrangian

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

  with the numerical factor included so that the equations of motion matches Maxwell's equations.

- We can express this as

$$F_{\mu\nu} = -\frac{i}{e}[D_\mu, D_\nu] = -\frac{i}{e}[\partial_\mu, \partial_\nu] + [\partial_\mu, A_\nu] + [A_\mu, \partial_\nu] + ie[A_\mu, A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Final Lagrangian density is

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m\right)\psi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$$

with $D_\mu = \partial_\mu + ieA_\mu$

- NOTE: no mass term for the photon such as $M^2 A_\mu A^\mu$. If we add this and make a $U(1)$ transformation

$$M^2 A_\mu A^\mu \rightarrow M^2 A'_\mu A'^\mu = M^2 A_\mu A^\mu - \frac{2M^2}{e}A^\mu\partial_\mu\omega$$
$$\rightarrow \delta\mathcal{L} = -\frac{2M^2}{e}A^\mu\partial_\mu\omega.$$

Thus gauge invariance requires a massless gauge boson.

2.2 Gauge Fixing

- In order to find the Feynman rule for the photon propagator (and to quantise the electromagnetic field) we look for the parts of the action which are quadratic in the photon field.

- E.g. for some field $\phi(x)$ we take fourier transforms of the fields and identify the terms

$$S_\phi = \int d^4p\tilde{\phi}(-p)\mathcal{O}(p)\tilde{\phi}(p),$$

with the propagator for ϕ being

$$-i\mathcal{O}^{-1}(p).$$

- In the case of QED let us derive from the action in x-space, we have

$$\begin{aligned} S &= \int d^4x\left[-\frac{1}{4}(F_{\mu\nu})^2\right] = \int d^4x\ (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= \int d^4x\ A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu \end{aligned}$$

Now use Fourier transform for A_μ as

$$A_\mu(x) = \int \frac{d^4p}{(2\pi)^4}\ \tilde{A}_\mu(p)\ e^{ipx}$$

$$\begin{aligned} S &= \int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\ \tilde{A}_\mu(p)\ e^{ipx}(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)\ \tilde{A}_\nu(k)\ e^{ikx} \\ &= \int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\ \tilde{A}_\mu(p)\ e^{i(p+k)x}(-k^2 g^{\mu\nu} - k^\mu k^\nu)\tilde{A}_\nu(k) \end{aligned}$$

$$= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\,\tilde{A}_\mu(p)\,\delta^4(p+k)(-k^2g^{\mu\nu}+k^\mu k^\nu)\tilde{A}_\nu(k)$$
$$= \int \frac{d^4p}{(2\pi)^4}\,\tilde{A}_\mu(p)\,(-p^2g^{\mu\nu}+p^\mu p^\nu)\tilde{A}_\nu(-p)$$

- However, this is not invertible, i.e.

$$(-p^2g^{\mu\nu}+p^\mu p^\nu)\tilde{D}_{\nu\rho}(p) = i\delta^\mu_\rho$$

has no solution as $(-p^2g^{\mu\nu}+p^\mu p^\nu)$ is singular. Cannot find the propagator.

- Issue really comes down to the fact that we have gauge invariance. Troublesome modes are those for which

$$A_\mu = \frac{1}{e}\partial_\mu\omega(x)$$

that is those which are gauge equivalent to

$$A_\mu(x) = 0.$$

Result is that the functional integral is badly defined...see QFT lectures

- Another interpretation is due to the fact that A_μ has four real components, introduced to maintain gauge symmetry. However the physical photon has two polarisation states.

- This difficulty can be resolved by fixing the gauge (breaking the gauge symmetry) in the Lagrangian in such a way as to maintain the gauge symmetry in observables.

- Can solve this using method presented by Faddeev and Popov - needs functional integrals to do properly but can be translated to adding to the Lagrangian density a gauge fixing term

$$-\frac{1}{2}\frac{(\partial.A)^2}{1-\xi}$$

leading to

$$S = \int \frac{d^4p}{(2\pi)^4}\,\tilde{A}_\mu(p)\,(-p^2g^{\mu\nu}+\frac{\xi}{1-\xi}p^\mu p^\nu)\tilde{A}_\nu(-p) \tag{2.1}$$

using the relation

$$\left(p^2g^{\mu\nu}-\frac{\xi}{1-\xi}p^\mu p^\nu\right)\left(g_{\nu\rho}-\frac{\xi p_\nu p_\rho}{p^2}\right) = p^2 g^\mu_\rho$$

we can now write the photon propagator as

$$-i\left(g_{\mu\nu}-\xi\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}$$

- Special choice of $\xi = 0$ is known as the Feynman gauge. In this gauge the propagator is particularly simple.
- We have to fix the gauge in order to be able to perform a calculation. Once we have computed a physical measurable quantity, the dependence on the gauge cancels.
- Taking $\xi = 0$ is one choice, there are other special choices but the choice does not affect the result of a calculation of a measurable quantity.

2.3 Summary - QED Lagrangian

Theory contains fermions with charge e, and a massless vector boson, the photon.

In Feynman Gauge

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi - \frac{1}{2}(\partial.A)^2$$

with $D_\mu = \partial_\mu + ieA_\mu$

Invariant under the transformations

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \omega\left(x\right)$$

$$\psi(x) \to \psi(x)' = e^{i\omega(x)}\psi(x) \qquad \bar{\psi}(x) \to \bar{\psi}(x)' = e^{-i\omega(x)}\bar{\psi}(x)$$

3 Non-Abelian Gauge Theories

3.1 Non-Abelian gauge transformations

- Extend "gauge" concept so that gauge bosons have self- interactions as observed for the gluons of QCD, and the $W^{\pm}$, Z of the electroweak sector.
- However, the gauge bosons will still be massless. (We will see how to give the $W^{\pm}$ and Z their observed masses in the Higgs chapter.)
- Non-Abelian means different elements of the group do not commute with each other.
- Transformations act between a degenerate space of states (for example between u-d quarks states where both have the same masses in isospin)
- Consider n free fermion fields ψ, arranged in a multiplet ψ

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ . \\ . \\ \psi_n \end{pmatrix}$$

- The Lagrangian density for a ψ is

$$\mathcal{L} \;=\; \overline{\psi}^{\,i}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi_i, \tag{3.1}$$

where the index i is summed from 1 to n. Eq.(3.1) is therefore shorthand for

$$\mathcal{L} \;=\; \overline{\psi}^{\,1}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi_1 \;+\; \overline{\psi}^{\,2}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi_2 + ... \tag{3.2}$$

- The Lagrangian density is invariant under the global complex transformations in field (ψ_i) space:

$$\psi \to \psi' = \mathbf{U}\psi \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\mathbf{U}^{\dagger}$$

where $\mathbf{U}$ is an $n \times n$ unitary matrix

$$\mathbf{U}^{\dagger}\mathbf{U} = 1, \qquad \det[\mathbf{U}] = 1$$

- These matrices will form the group $SU(n)$.

- In order to specify an $SU(n)$ matrix we need $n^2 - 1$ real parameters.
- An arbitrary $SU(n)$ matrix can be written as

$$\mathbf{U} = e^{-i\Sigma_{a=1}^{n^2-1}\omega^a \mathbf{T^a}} \equiv e^{-i\omega^a \mathbf{T^a}}$$

- ω^a are real parameters, and the $\mathbf{T^a}$ are the generators of the $SU(n)$ group
- Generators are hermitian and traceless (can prove these facts by unitarity condition and determinant condition respectively
- For $SU(n)$ there are n^2-1 generators $\mathbf{T^a}$ which are normalised, in the fundamental representation, using the convention

$$\mathrm{tr}(\mathbf{T^a T^b}) = \frac{1}{2}\delta_{ab}$$

- Two such transformations do *not* commute.

$$\left(e^{i\omega_1^a \mathbf{T}^a}\right)\left(e^{i\omega_2^b \mathbf{T}^b}\right) \neq \left(e^{i\omega_2^b \mathbf{T}^b}\right)\left(e^{i\omega_1^a \mathbf{T}^a}\right).$$

- Specifically the Lie algebra of the $SU(n)$ group is written in terms of commutators:

$$[\mathbf{T}^a, \mathbf{T}^b] = if^{abc}\mathbf{T}^c$$

where f^{abc} are called structure functions of the $SU(n)$ group and are totally antisymmetric.

3.2 Non-Abelian Gauge Fields

- Now consider local $SU(n)$ transformations, as with abelian symmetries the lagrangian is no longer invariant

$$\mathcal{L} \to \mathcal{L}' = \bar{\psi}\mathbf{U}^\dagger(i\gamma^\mu\partial_\mu - m)\mathbf{U}\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + i\bar{\psi}\mathbf{U}^\dagger\gamma^\mu(\partial_\mu\mathbf{U})\psi$$

- In analogy with the abelian case we introduce the covariant derivative

$$\mathbf{D}_\mu = \partial_\mu + ig\mathbf{A}_\mu$$

where $\mathbf{A}_\mu = \mathbf{T}^a A_\mu$

- Covariant derivative contains $n^2 - 1$ (spin 1) gauge bosons, A_μ^a, one for each generator.

- The quantity $\mathbf{A}_\mu$ transforms as

$$\mathbf{A}_\mu \to \mathbf{A}'_\mu = \mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger + \frac{i}{g}(\partial_\mu \mathbf{U})\mathbf{U}^\dagger \tag{3.3}$$

- sometimes useful to use infinitesimal form which can be found by the following, using $\mathbf{U} = e^{-i\omega^a T^a}$,

$$\begin{aligned}
\mathbf{A}'_\mu &= \mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger + \frac{i}{g}(\partial_\mu \mathbf{U})\mathbf{U}^\dagger = e^{-i\omega^b\mathbf{T}^b} A^a_\mu \mathbf{T}^a e^{i\omega^b\mathbf{T}^b} + \frac{i}{g}(\partial_\mu e^{-i\omega^a\mathbf{T}^a})e^{i\omega^a\mathbf{T}^a} \\
&\approx (1 - i\omega^b\mathbf{T}^b)A^a_\mu\mathbf{T}^a(1 + i\omega^b\mathbf{T}^b) + \frac{i}{g}(\partial_\mu(1 - i\omega^a\mathbf{T}^a))(1 + i\omega^a\mathbf{T}^a) \\
&\approx A^a_\mu\mathbf{T}^a - i\omega^b\mathbf{T}^b A^a_\mu\mathbf{T}^a + iA^a_\mu\mathbf{T}^a\omega^b\mathbf{T}^b + \frac{1}{g}(\partial_\mu\omega^a)\mathbf{T}^a \\
&\approx A^a_\mu\mathbf{T}^a + i[\mathbf{T}^a, \mathbf{T}^b]\omega^b A^a_\mu + \frac{1}{g}(\partial_\mu\omega^a)\mathbf{T}^a \\
&\approx A^a_\mu\mathbf{T}^a - f^{abc}\mathbf{T}^c\omega^b A^a_\mu + \frac{1}{g}(\partial_\mu\omega^a)\mathbf{T}^a
\end{aligned}$$

- It is easy to check how $D_\mu\psi$ transforms:

$$\begin{aligned}
\mathbf{D}_\mu\psi \to (\mathbf{D}_\mu\psi)' &= \left[\partial_\mu + ig(\mathbf{U}\mathbf{A}_\mu\mathbf{U}^\dagger + \frac{i}{g}(\partial_\mu\mathbf{U})\mathbf{U}^\dagger\right]\mathbf{U}\psi \\
&= \mathbf{U}\partial_\mu\psi + ig\mathbf{U}\mathbf{A}_\mu\psi - (\partial_\mu\mathbf{U})\psi + (\partial_\mu\mathbf{U})\psi \\
&= \mathbf{U}\partial_\mu\psi + ig\mathbf{U}\mathbf{A}_\mu\psi \\
&= \mathbf{U}\mathbf{D}_\mu\psi
\end{aligned} \tag{3.4}$$

- Trivial to then check invariance of the lagrangian

- The kinetic term for the gauge bosons is again constructed from the field strengths $F^a_{\mu\nu}$ which are defined from the commutator of two covariant derivatives:

$$\mathbf{F}_{\mu\nu} = -\frac{i}{g}\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right]. \tag{3.5}$$

where the matrix $\mathbf{F}_{\mu\nu}$ is given by

$$\mathbf{F}_{\mu\nu} = \mathbf{T}^a F^a_{\mu\nu},$$

$$\begin{aligned}
\mathbf{F}_{\mu\nu} = -\frac{i}{g}\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right] &= (\partial_\mu + ig\mathbf{A}_\mu)(\partial_\nu + ig\mathbf{A}_\nu) - (\partial_\nu + ig\mathbf{A}_\nu)(\partial_\mu + ig\mathbf{A}_\mu) \\
&= (\partial_\mu\mathbf{A}_\nu) - (\partial_\nu\mathbf{A}_\mu) + ig(\mathbf{A}_\mu\mathbf{A}_\nu - \mathbf{A}_\nu\mathbf{A}_\mu)
\end{aligned}$$

$$
\begin{aligned}
&= (\partial_\mu A^a_\nu)\mathbf{T}^a - (\partial_\nu A^a_\mu)\mathbf{T}^a + igA^b_\mu A^c_\nu(\mathbf{T}^b\mathbf{T}^c - \mathbf{T}^c\mathbf{T}^b) \\
&= [(\partial_\mu A^a_\nu) - (\partial_\nu A^a_\mu)]\mathbf{T}^a + igA^b_\mu A^c_\nu[\mathbf{T}^b, \mathbf{T}^c] \\
&= \left([(\partial_\mu A^a_\nu) - (\partial_\nu A^a_\mu)] - gf^{abc}A^b_\mu A^c_\nu\right)\mathbf{T}^a
\end{aligned}
$$

- This gives us

$$
F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc}\, A^b_\mu A^c_\nu \tag{3.6}
$$

- From (3.3) we can deduce how $\mathbf{F}_{\mu\nu}$ transforms

$$
\mathbf{F}_{\mu\nu} \to -\frac{i}{g}([\mathbf{D}_\mu, \mathbf{D}_\nu])' = -\frac{i}{g}\mathbf{U}\,[\mathbf{D}_\mu, \mathbf{D}_\nu]\,\mathbf{U}^\dagger
$$

- Easy to show that the infinitesimal version is

$$
\mathbf{F}_{\mu\nu} \to (\mathbf{F}_{\mu\nu})' = F^a_{\mu\nu}\mathbf{T}^a + f^{abc}F^b_{\mu\nu}\omega^c\mathbf{T}^a
$$

- The gauge invariant term giving the gauge boson kinetic terms is:

$$
-\frac{1}{2}\mathrm{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right] = -\frac{1}{2}F^a_{\mu\nu}F^{b\,\mu\nu}\mathrm{Tr}\left[\mathbf{T}^a\mathbf{T}^b\right] = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}
$$

- In contrast to the Abelian case, this term contains quadratic pieces in the derivatives of the gauge boson fields, but also interaction terms:

$$
\begin{aligned}
-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} \;&=\; [(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g\, f^{abc}\, A^b_\mu A^c_\nu][(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - g\, f^{ade}\, A^{d\mu} A^{e\nu}] \qquad (3.7) \\
&= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - g\, f^{abc}\, A^b_\mu A^c_\nu(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\
&- \; g\, f^{ade}\, A^{d\mu} A^{e\nu}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + g^2\, f^{abc} f^{ade}\, A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}
\end{aligned}
$$

- For non-Abelian gauge theories the gauge bosons interact with each other via both three-point and four-point interaction terms.

- Once again, a mass term for the gauge bosons is forbidden, since a term proportional to $A^a_\mu A^{a\,\mu}$ is *not* gauge invariant.

3.3 Gauge Fixing

- As with the abelian symmetries, we need to add a gauge fixing term in order to be able to derive a propagator for the gauge bosons.
- In the Feynman gauge this means adding the term

$$-\frac{1}{2}(\partial^\mu A^a_\mu)^2$$

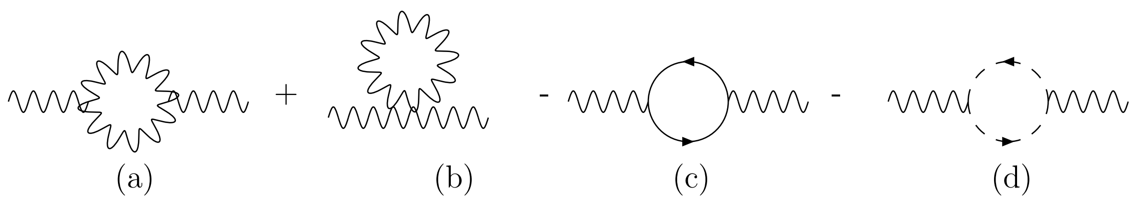
to the Lagrangian density and the propagator (in momentum space) becomes

$$-i\,\delta_{ab}\,\frac{g_{\mu\nu}}{p^2}.$$

- One unfortunate complication only needed for the purpose of performing higher loop calculations with non-Abelian gauge theories.
- A consequence of gauge-fixing is that we need extra loop diagrams in higher order which are mathematically equivalent to interacting scalar particles known as a "Faddeev-Popov ghost" for each gauge field.
- These are not physical scalar particles and cannot appear as external legs on diagrams

1. They only occur inside loops.
2. They behave like fermions even though they are scalars (spin zero). This means that we need to count a minus sign for each loop of Faddeev-Popov ghosts in any Feynman diagram.

For example, the Feynman diagrams which contribute to the one-loop corrections to the gauge boson propagator are



- (a) - three-point interaction between the gauge bosons
- (b)- four-point interaction between the gauge bosons
- (c) involves a loop of fermions,
- (d) extra diagram involving the Faddeev-Popov ghosts.
- Both (c) and (d) have a minus sign in front of them because both the fermions and Faddeev-Popov ghosts obey Fermi statistics.

3.4 Summary: The Lagrangian for a General non-Abelian Gauge Theory

Consider a group gauge group, $\mathcal{G}$ of "dimension" N, whose N generators, $\mathbf{T}^a$, obey the commutation relations

$$\left[\mathbf{T}^a, \mathbf{T}^b\right] \;=\; i f^{abc}\mathbf{T}^c, \tag{3.8}$$

where f_{abc} are called the "structure constants" of the group (they are antisymmetric in the indices a, b, c).

- These matrices satisfy the Jacobi identity,

$$\left[\mathbf{T}^A, \left[\mathbf{T}^B, \mathbf{T}^C\right]\right] + \left[\mathbf{T}^B, \left[\mathbf{T}^C, \mathbf{T}^A\right]\right] + \left[\mathbf{T}^C, \left[\mathbf{T}^A, \mathbf{T}^B\right]\right] = 0 \tag{3.9}$$

  which means,

$$f^{ADE} f^{BCD} + f^{BDE} f^{CAD} + f^{CDE} f^{ABD} = 0. \tag{3.10}$$

  We can then define a $d_a \times d_a$,matrix,

$$(T^C)_{AB} \equiv -i f_{ABC}, \tag{3.11}$$

  which satisfy the Lie algebra.

- The matrices form a real representation in a d_a-dimensional vector space and is called the adjoint. The dimension of this space is equal to the number of generators.

- The ${\mathbf{T}_\mathbf{r}}^A$ matrices which represent the algebra in representation $\mathbf{r}$ satisfy a normalisation condition,

$$\mathrm{Tr}({\mathbf{T}_\mathbf{r}}^A {\mathbf{T}_\mathbf{r}}^B) = C_\mathbf{r} \delta^{AB}. \tag{3.12}$$

  where $C_\mathbf{r}$ is the Dynkin index for the representation $\mathbf{r}$.

- In addition if we multiply by δ^{AB} and sum over A and B, we have,

$$d_\mathbf{r} C(\mathbf{r}) = C_\mathbf{r} d_a \tag{3.13}$$

  where $d_\mathbf{r}$ is the dimension of the representation and $C(\mathbf{r})$ is the quadratic Casimir operator of the representation $\mathbf{r}$, defined by

$$\sum_A^{d_a} {\mathbf{T}_\mathbf{r}}^A {\mathbf{T}_\mathbf{r}}^B = C(\mathbf{r})\mathbf{I}$$

  where $\mathbf{I}$ is the $d_\mathbf{r} \times d_\mathbf{r}$ identity matrix.

- In Table 1, we summarise some properties of the fundamental and adjoint representations of $SU(N)$.

Representation	$d_{\mathbf{r}}$	$C_{\mathbf{r}}$	$C(\mathbf{r})$
Fundamental	N	$1/2$	$(N^2-1)/2N$
Adjoint	N^2-1	N	N

Table 1: In this Table we list various properties for the $SU(N)$ symmetry group.

The Lagrangian density for a gauge theory with this group, with a fermion multiplet ψ_i is given (in Feynman gauge) by

$$\mathcal{L} \;=\; -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + i\overline{\psi}\left(\gamma^\mu \mathbf{D}_\mu - m\mathbf{I}\right)\psi - \frac{1}{2}(\partial^\mu A^a_\mu)^2 + \mathcal{L}_{FP} \tag{3.14}$$

where

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc} A^b_\mu A^c_\nu, \tag{3.15}$$

$$\mathbf{D}_\mu \;=\; \partial_\mu \mathbf{I} + i\, g\, \mathbf{T}^a A^a_\mu \tag{3.16}$$

and

$$\mathcal{L}_{FP} \;=\; -\xi^a \partial^\mu \partial_\mu \eta^a + g\, f_{acb} \xi^a A^c_\mu (\partial^\mu \eta^b)$$

Under an infinitesimal gauge transformation, the N gauge bosons, A^a_μ change by an amount that contains a term which is not linear in A^a_μ:

$$\delta A^a_\mu(x) \;=\; f^{abc} A^b_\mu(x)\omega^c(x) - \frac{1}{g}\partial_\mu \omega^a(x), \tag{3.17}$$

whereas the field strengths $F^a_{\mu\nu}$ transform by a change

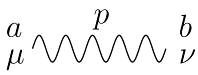
$$\delta F^a_{\mu\nu}(x) \;=\; f^{abc}\, F^b_{\mu\nu}(x)\, \omega^c. \tag{3.18}$$

- We say that the Gauge bosons transform as the "adjoint" representation of the group (which has as many components as there are generators).

3.5 Feynman Rules

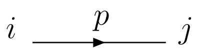
The Feynman rules for such a gauge theory are given by: **Propagators:**

Gauge Boson



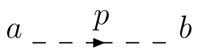
$$-i\,\delta_{ab} g_{\mu\nu}/p^2$$

Fermion



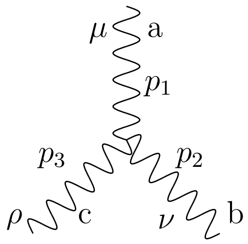
$$i\,\delta_{ij}(\gamma^\mu p_\mu + m)/(p^2 - m^2)$$

Faddeev-Popov ghost

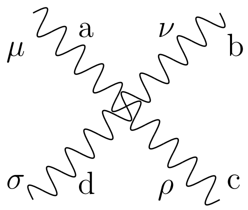


$$i\,\delta_{ab}/p^2$$

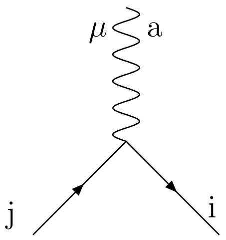
Vertices:
(all momenta are flowing into the vertex).



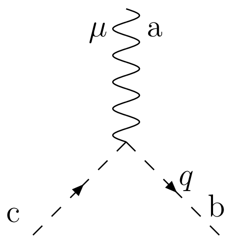
$$-g\, f_{abc} \Big(g_{\mu\nu} \left(p_1 - p_2\right)_\rho + g_{\nu\rho} \left(p_2 - p_3\right)_\mu + g_{\rho\mu} \left(p_3 - p_1\right)_\nu \Big)$$



$$-i\, g^2 f_{eab} f_{ecd} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$
$$-i\, g^2 f_{eac} f_{ebd} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$
$$-i\, g^2 f_{ead} f_{ebc} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}\right)$$



$$-i\, g\, \gamma^\mu \left(T^a\right)_{ij}$$



$$g\, f_{abc}\, q_\mu$$

4 Quantum Chromodynamics

4.1 Colour and the Spin-Statistics Problem

- Motivation for colour:

$$\Delta^{++}$$

consists of uuu and has spin 3/2 and is in the $l = 0$ angular momentum state
wavefunction seems to be symmetric under exchange of quarks - exclusion principle says this cannot be so

- Fermions should have a wavefunction which is *antisymmetric* under the interchange of the quantum numbers of any two fermions.

- Solved by allowing quarks to also carry one of three "colours"

- As well as the flavour index, (f) a quark field carries a colour index, i, so we write a quark field as $q^i_f, \ i = 1 \cdots 3$.

- It was further assumed that the strong interactions are invariant under colour SU(3) transformations.

- The quarks transform as a triplet representation of the group $SU(3)$ which has eight generators.

- Furthermore it was assumed that all the observed hadrons are singlets of this new SU(3) group.

- Spin statistics problem is now solved

- Baryon consists of three quarks and a colour singlet state

- Colour part of the wavefunction is

$$|B> = \ \epsilon_{ijk} |q^i_{f_1} q^j_{f_2} q^k_{f_3}>,$$

where f_1, f_2, f_3 are the flavours of the three quarks that make up the baryon B and i, j, k are the colours.

- Tensor ϵ_{ijk} is totally antisymmetric under the interchange of any two indices, so that if the part of the wavefunction of the baryon that does *not* depend on colour is symmetric, the total wavefunction (including the colour part) is antisymmetric, as required by the spin-statistics theorem.

4.2 QCD

- QCD is where invariance under colour SU(3) transformations is promoted to an invariance under local SU(3) (gauge) transformations.
- Quarks transform as triplets and anti-quarks transform as anti-triplets under $SU(3)_C$
- a quark transforms as
$$q \to q' = e^{\frac{i}{2}\boldsymbol{\alpha}.\boldsymbol{\lambda}} q$$
with eight finite real parameters ($\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_8)$) and $\boldsymbol{\lambda}$ stands for eight Hermitian 3×3 matrices (the generators of SU(3)), $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_8)$,

- These are the *Gell-Mann matrices. These eight Hermitian* 3×3 *matrices represent the generators of SU(3) in the fundamental representation.*

$$\begin{array}{lll}
\lambda_1 = \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 = \begin{pmatrix} \mathbf{0} & -\boldsymbol{i} & 0 \\ \boldsymbol{i} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & -\mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix} &
\end{array} \tag{4.1}$$

- The eight gauge bosons which have to be introduced in order to preserve this invariance are the eight "gluons".
- These are taken to be the carriers which mediate the strong interactions in exactly the same way that photons are the carriers which mediate the electromagnetic interactions.

- The Feynman rules for QCD are therefore simply the Feynman rules listed in the previous lecture, with, g, taken to be the strong coupling, g_s, (more about this later)
- The generators $\mathbf{T}^a = \frac{1}{2}\lambda^a$ taken to be the eight generators above with algebra
$$[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \mathbf{T}^c$$

with $f_{abc},\ a,\ b,\ c, = 1\cdots 8$ the structure constants of SU(3) with values

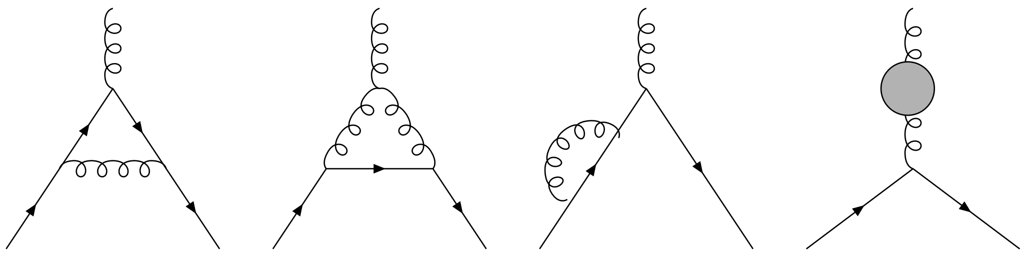
$$\begin{aligned} f^{123} &= 1 \\ f^{147} &= -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} \\ f^{458} &= f^{678} = \frac{\sqrt{3}}{2} \end{aligned}$$

- To maintain gauge invariance we again exchange $\partial_\mu \to D_\mu = \partial_\mu - \frac{ig_s}{2} A^a_\mu \lambda^a$

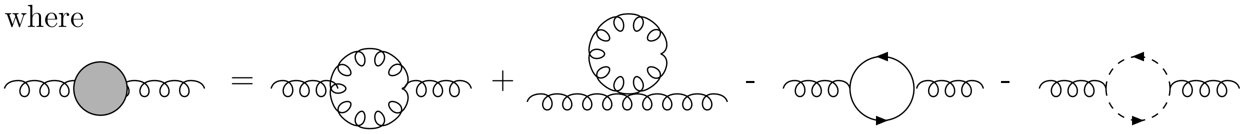
Thus we now have a Quantum Field Theory which can be used to describe the strong interactions.

4.3 Running Coupling

- The coupling for the strong interactions is the QCD gauge coupling, g_s. We usually work in terms of α_s defined as
$$\alpha_s = \frac{g_s^2}{4\pi}.$$
- Since the interactions are strong, we would expect α_s to be too large to perform reliable calculations in perturbation theory.
- On the other hand the Feynman rules are only useful within the context of perturbation theory.
- Difficulty resolved as "coupling constants" are not constant at all.
- The electromagnetic fine structure constant, α, only has the value 1/137 at energies which are not large compared with the electron mass.
  At higher energies it is larger than this.
- For example, at LEP energies it takes a value closer to 1/128.
- On the other hand, it turns out that for non-Abelian gauge theories the coupling *decreases* as the energy increases.
- To see how in QCD, we note that when we perform higher order perturbative calculations there are loop diagrams which dress the couplings.
- For example, the one-loop diagrams which dress the coupling between a quark and a gluon are:



where



are the diagrams needed to calculate the one-loop corrections to the gluon propagator.

- These diagrams contain UV divergences and need to be renormalised by subtracting at some renormalisation scale μ.
- This scale then appears inside a logarithm for the renormalised quantities.
- This means that if the square-momenta of all the external particles coming into the vertex are of order Q^2, where $Q \gg \mu$, then the above diagrams give rise to a correction which contains a logarithm of the ratio Q^2/μ^2:
$$-\alpha_s^2 \, \beta_0 \, \ln\left(Q^2/\mu^2\right).$$
- This correction is interpreted as the correction to the effective QCD coupling, $\alpha_s(Q^2)$, at momentum scale Q.
- A calculation shows that the effective coupling obeys the differential equation
$$\frac{\partial \, \alpha_s(Q^2)}{\partial \, \ln(Q^2)} \;=\; \beta\left(\alpha_s(Q^2)\right) \tag{4.2}$$
where β has a perturbative expansion
$$\beta(\alpha) \;=\; -\beta_0 \, \alpha^2 + \mathcal{O}(\alpha^3) + \cdots \tag{4.3}$$
where β_0 is calculated to be
$$\beta_0 \;= \frac{1}{4\pi}\left[\frac{11}{3}C_{adj} - \frac{2}{3}\sum_i C_{s_i} - \frac{1}{6}\sum_a C_{r_a}\right], \tag{4.4}$$

where where considering contributions from fields transforming as the adjoint, scalars transforming with representation s_i and fermions transforming with representation r_a. The Cs are the dynkin indices for the particular representations, r, defined by

$$Tr[T_r^a T_r^b] = C_r \delta^{ab} \tag{4.5}$$

- There are no scalars charged under QCD
- For the adjoint and fermions transforming under the fundamental and anti-fundamental we have

$$Tr[T_{adj}^a T_{adj}^b] = N_c \delta^{ab}, \qquad Tr[T_F^a T_F^b] = \frac{1}{2}\delta^{ab}, \qquad Tr[T_{\bar{F}}^a T_{\bar{F}}^b] = \frac{1}{2}\delta^{ab} \tag{4.6}$$

- So we get with $N_c = 3$

$$\beta_0 = \frac{1}{4\pi}\left[\frac{11}{3}3 - \frac{2}{3}\left[n_f\frac{1}{2} + n_f\frac{1}{2}\right]\right] = \frac{[33 - 2n_f]}{12\pi}$$

- n_f is the number of active flavours, i.e. the number of flavours whose mass threshold is below the momentum scale, Q.
- Solving this **do it from the differential form** we find

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s(\mu^2) \ln \frac{Q^2}{\mu^2}}$$

  Now we need a boundary value.

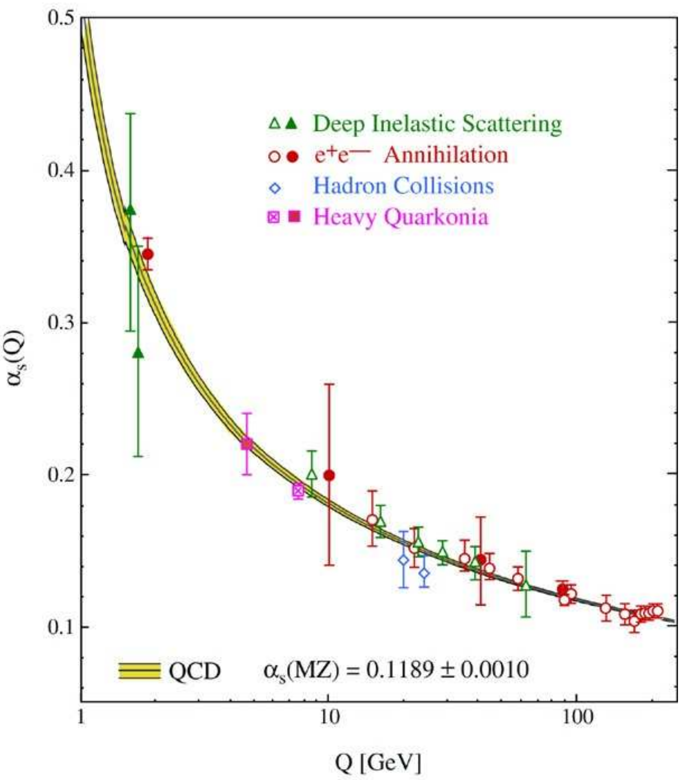
- This is taken to be the measured value of the coupling at the $Z-$boson mass ($= 91$ GeV), which is measured to be

$$\alpha_s(M_Z^2) \;=\; 0.118 \pm 0.002 \tag{4.7}$$

  This is one of the free parameters of the standard model and is what we use to replace μ and $\alpha(\mu^2)$.

- Note that β_0 is *positive*, which means that the effective coupling *decreases* as the momentum scale is increased.

- The running of $\alpha_s(Q^2)$ is shown in the figure.
- We can see that for momentum scales above about 2 GeV the coupling is less than 0.3 so that one can hope to carry out reliable perturbative calculations for QCD processes with energy scales larger than this.



- Gauge invariance requires that the gauge coupling for the interaction between gluons must be exactly the same as the gauge coupling for the interaction between quarks and gluons.

- The $\beta-$function could therefore have been calculated from the higher order corrections to the three-gluon (or four-gluon) vertex
$\rightarrow$ must yield the same result, despite the fact that it is calculated from a completely different set of diagrams.

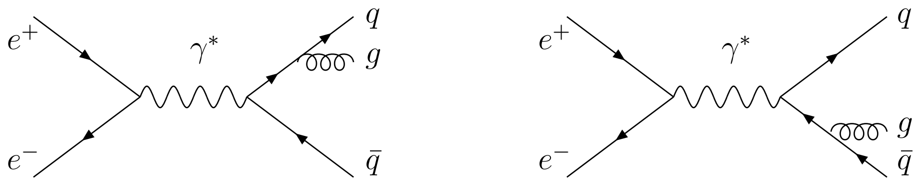
4.4 Quark (and Gluon) Confinement

- This argument can be inverted to provide an answer to the question 'why have we never seen quarks or gluons in a laboratory ? '.

- Asymptotic freedom, which tells us that the **effective coupling between quarks becomes weaker as we go to short distances** (this is equivalent to going to high energies) implies, conversely, that **effective couplings grow as we go to large distances**.

- Therefore, the complicated system of gluon exchanges, which leads to the binding of quarks (and antiquarks) inside hadrons, leads to a stronger and stronger binding as we attempt to pull the quarks apart.
- This means that we can never isolate a quark (or a gluon) at large distances since we require more and more energy to overcome the binding as the distance between the quarks grows.
- Only free particles which can be observed at macroscopic distances from each other are colour singlets.
- Mechanism is known as "quark confinement".
- Details are not fully understood.
  At the level of non-perturbative field theory, lattice calculations have confirmed that the binding energy grows as the distance between quarks increases.
- Thus we have two different pictures of the world.
- **Short distances**, or **large energies**, quarks and gluons are the appropriate degrees of freedom to do calculations with.
  They are what we consider interacting with each other.
  e.g. We can perform calculations of the scattering cross-sections between quarks and gluons (called the "hard cross-section")
- This is because running coupling is sufficiently small so that we can rely on perturbation theory.
- However, on the other hand in experiments, we need to take into account the fact that these quarks and gluons bind into colour singlet hadrons
  $\rightarrow$ only these colour singlet states that are observed.
- The mechanism for such binding is beyond the scope of perturbation theory and is not understood in detail.
- Monte Carlo programs have been developed which simulate this binding in such a way that the results of the short-distance perturbative calculations at the level of quarks and gluons can be confronted with experiment in a successful way.
- Thus, for example, to calculate the cross-section for electron-positron annihilation into three jets (at high energies)
  First calculate, in perturbation theory, the process for electron plus positron to annihilate into a virtual photon which then decays into a quark, and antiquark and a gluon.

- The two Feynman diagrams for this process are:



- However, before we can compare with experimental data we need to perform a convolution of this calculated cross-section with a Monte Carlo.
  This simulates the way in which the final state partons (quarks and gluons) bind with other quarks and gluons to produce observed hadrons.

- It is only after such a convolution has been performed that one can get a reliable comparison of the calculated cross-section with data.

- Likewise, if we want to calculate cross-sections for initial state hadrons we need to account for the probability of finding a particular quark or gluon inside an initial hadron with a given fraction of the initial hadron's momentum (these are called "parton distribution functions").

More later on the structure of hadrons.

4.5 $\theta-$ Parameter and Strong CP problem.

- There is one more gauge invariant term that can be added to the Lagrangian density for QCD. This term is
$$\mathcal{L}_\theta \;=\; \theta \frac{\alpha_s}{8\pi} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \;=\; \theta \frac{\alpha_s}{8\pi} F^a_{\mu\nu} \tilde{F}^{a\mu\nu},$$
  where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor (in four dimensions) and $\tilde{F}$ is the dual field strength.

- Such a term arises when one considers "instantons" (which are beyond the scope of these lectures.)

- This term violates CP. In QED we would have
$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \;=\; \mathbf{E}\cdot\mathbf{B},$$
  and for QCD we have a similar expression except that $\mathbf{E}^a$ and $\mathbf{B}^a$ carry a colour index - they are known as the chromoelectric and chromomagnetic fields.

- Under charge conjugation both the electric and magnetic filed change sign, but under parity the electric field, which is a proper vector, changes sign, whereas the magnetic field, which is a polar vector, does not change sign.

- Thus we see that the term $\mathbf{E} \cdot \mathbf{B}$ is odd under CP.

- For this reason, the parameter θ in front of this term must be exceedingly small in order not to give rise to strong interaction contributions to CP violating quantities such as the electric dipole moment of the neutron.

- The current experimental limits on this dipole moment tell us that $\theta < 10^{-9}$ and it is probably zero.

- Nevertheless, strictly speaking θ is a free parameter of QCD, and is often considered to be the nineteenth free parameter of the Standard Model.

- Of course we simply could set $\theta = 0$ (or a very small number), however term is regenerated through loops

- Even if we could set it to zero we want to know why.

- The fact that we do not know why this term is absent (or so small) is the strong CP problem.

- Several possible solutions to the strong CP problem that offer explanations as to why this term is absent (or small).

- One possible solution: add additional symmetry, leading to the postulation of a new, hypothetical, weakly interacting particle, called the (Peccei-Quinn) axion.

- Unfortunately none of these solutions have been confirmed yet and the problem is still unresolved.

- Another question: why no problem in QED?

- This term can be written (in QED and QCD) as a total divergence, so it seems that it can be eliminated from the Lagrangian altogether.

- However, in QCD (but not in QED) there are non-perturbative effects from the non-trivial topological structure of the vacuum (somewhat related to so called instantons you probably have heard about) which prevent us from neglecting the θ-term

4.6 Summary

- Quarks transform as a triplet representation of colour SU(3) (each quark can have one of three colours.)
- The requirement that the observed hadrons must be singlets of colour SU(3) solves the spin-statistics problem for baryons. The wavefunction for a colour singlet state of three quarks has an antisymmetric colour component.
- QCD is the SU(3) gauge theory in which the symmetry under colour SU(3) is taken to be local.
- The eight gauge bosons of QCD are the gluons which are the carriers that mediate the strong interactions. They are massless.
- The coupling of quarks to gluons (and gluons to each other) decreases as the energy scale increases. Therefore, at high energies one can perform reliable perturbative calculations for strong interacting processes.
- As the distance between quarks increases the binding increases, such that it is impossible to isolate individual quarks or gluons.
  $\rightarrow$ The only observable particles are colour singlet hadrons.
- Perturbative calculations performed at the quark and gluon level must be modified by accounting for the recombination of final state quarks and gluons into observed hadrons as well as the probability of finding these quarks and gluons inside the initial state hadrons.
- QCD admits a gauge invariant strong CP violating term with a coefficient θ. This parameter is known to be very small from limits on CP violating phenomena such as the electric dipole moment of the neutron.

5 Spontaneous Symmetry Breaking

- We have seen that in an unbroken gauge theory the gauge bosons must be massless.
- The only observed massless spin-1 particles are photons. In the case of QCD, the gluons are also massless, but they cannot be observed because of confinement.
- To extend the ideas of describing interactions by a gauge theory to the weak interactions, the symmetry must somehow be broken since the carriers of weak interactions (W- and Z-bosons) are massive (weak interactions are very short range).
- We could simply break the symmetry by hand by adding a mass term for the gauge bosons, which we know violates the gauge symmetry.
- However, this would destroy renormalizability of our theory.
- Renormalizable theories are preferred because they are more predictive.

5.1 Massive gauge bosons and Renormalisability

- Show in a little more detail how explicit breaking mean non-renormalisability
- Higher order (loop) corrections generate ultraviolet divergences.
- In a renormalisable theory, these divergences can be absorbed into the parameters of the theory and in this way can be 'hidden'.
- As we go to higher orders we need to absorb more and more terms into these parameters, but there are only as many divergent quantities as there are parameters.
- E.g. in the QED Lagrangian we have a fermion field, the gauge boson field and interactions whose strength is controlled by e and m
- All divergences of diagrams can be absorbed into these quantities
- Once measured all other predictions can be written in terms of these parameters
- There are conditions on allowed terms in a renormalisable theory
- Furthermore all the propagators have to decrease like $1/p^2$ as the momentum $p \to \infty$.

- If conditions are not fulfilled then the theory generates more and more divergent terms as one calculates to higher orders and it is not possible to absorb these divergences into the parameters of the theory.

- Such theories are said to be “non-renormalisable”.

- So how does adding an explicit mass term $M^2 A_\mu A^\mu$ ruin renormalisabililty?

- This term modifies the propagator,

$$\frac{1}{2} A_\mu \left(-g^{\mu\nu}(p^2 - M^2) + p^\mu p^\nu \right) A_\nu$$

  inverting this we have

$$\frac{i}{p^2 - M^2} \left(g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \right)$$

  note that this propagator has a much worse UV behaviour, it goes to a constant as $p \to \infty$. compared to

$$-i \left(g^{\mu\nu} - \xi \frac{p^\mu p^\nu}{p^2} \right) \frac{1}{p^2}$$

- With the explicit mass term the theory has more UV divergences

- It is the gauge symmetry that ensures renormalisabililty

- There is a far more elegant way of doing this which is called “spontaneous symmetry breaking”.

- In this scenario, the Lagrangian maintains its symmetry under a set of local gauge transformations.

- On the other hand, the lowest energy state, which we interpret as the vacuum state, is *not* a singlet of the gauge symmetry.

- There is an infinite number of states each with the same ground-state energy and Nature chooses one of these states as the ‘true’ vacuum.

5.2 Spontaneous Symmetry Breaking

- Spontaneous symmetry breaking is a phenomenon that is not restricted to gauge symmetries.

- In order to illustrate the idea of spontaneous symmetry breaking:
  Consider a pen that is completely symmetric with respect to rotations around its axis.

- If we balance this pen on its tip on a table, and start to press on it with a force precisely along the axis we have a perfectly symmetric situation.
- This corresponds to a Lagrangian which is symmetric (under rotations around the axis of the pen in this case).
- However, if we increase the force, at some point the pen will bend (and eventually break).
- The question then is in which direction will it bend. Of course we do not know, since all directions are equal.
- But the pen will pick one and by doing so it will break the rotational symmetry. This is spontaneous symmetry breaking.

- Better example can be given by looking at a point mass in a potential

$$V(r) = \mu^2 \bar{r}.\bar{r} + \lambda(\bar{r}.\bar{r})^2$$

- Potential is symmetric under rotations and we assume $\lambda > 0$ (otherwise there would be no stable ground state).
- For $\mu^2 > 0$ potential has a minimum at $r = 0$, thus the point mass will simply fall to this point.
- The situation is more interesting if $\mu^2 < 0$.
- For two dimensions the potential is a mexican hat potential.
- If the point mass sits at $r = 0$ the system is not in the ground state and is not stable but the situation is completely symmetric.
- In order to reach the ground state, the symmetry has to be broken, i.e. if the point mass wants to roll down, it has to decide in which direction.
- Any direction is equally good, but one has to be picked.
- This is exactly what spontaneous symmetry breaking means.
- The Lagrangian (here the potential) is symmetric (here under rotations around the z-axis), but the ground state (here the position of the point mass once it rolled down) is not.
- Mathematically: Ground state is $|0\rangle$.

- A spontaneously broken gauge theory is a theory whose Lagrangian is invariant under gauge transformations but ground state is not invariant under gauge transformations.

$$e^{-i\omega^a \mathbf{T^a}}|0\rangle \neq 0$$

which means

$$T^a|0\rangle \neq 0 \qquad \text{for some } a$$

- Thus the theory is spontaneously broken if at least one generator does not annihilate the vacuum

5.3 Spontaneous Symmetry Breaking

- We start by considering a complex scalar field theory with a mass term and a quartic self-interaction.

- The Lagrangian density for such a theory may be written

$$\mathcal{L} \;=\; \partial_\mu \Phi^* \, \partial^\mu \Phi - V(\Phi), \tag{5.1}$$

where the "potential" $V(\Phi)$, is given by

$$V(\Phi) \;=\; \mu^2 \Phi^* \Phi + \lambda \left| \Phi^* \Phi \right|^2 . \tag{5.2}$$

- This Lagrangian is invariant under **global** $U(1)$ transformations

$$\Phi \;\rightarrow\; e^{i\omega}\Phi.$$

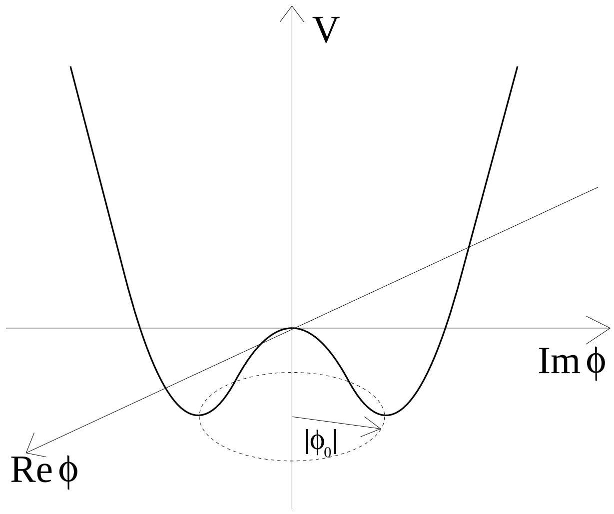
- Provided μ^2 is positive this potential has a minimum at $\Phi = 0$.

- We call the $\Phi = 0$ state the vacuum and expand Φ in terms of creation and annihilation operators that populate the higher energy states.

- In terms of a Quantum Field Theory, where Φ is an operator, the precise statement is that the operator, Φ, has zero "vacuum expectation value".

- Suppose now that we reverse the sign of μ^2, so that the potential becomes

$$V(\Phi) \;=\; -\mu^2 \Phi^* \Phi + \lambda \left| \Phi^* \Phi \right|^2 . \tag{5.3}$$



- We see that this potential no longer has a minimum at $\Phi = 0$, but a *maximum*. The minimum occurs at

$$\Phi \;=\; \frac{v}{\sqrt{2}} \;=\; e^{i\theta}\sqrt{\frac{\mu^2}{2\lambda}}, \tag{5.4}$$

where θ takes any value from 0 to 2π.

- There is an infinite number of states each with the same lowest energy - i.e. we have a degenerate vacuum.

- The symmetry breaking occurs in the choice made for the value of θ which represents the true vacuum.

- For convenience we shall choose $\theta = 0$ to be this vacuum.

- Such a choice constitutes a spontaneous breaking of the $U(1)$ invariance, since a $U(1)$ transformation takes us to a different lowest energy state.

- In other words the vacuum breaks $U(1)$ invariance.

- In Quantum Field Theory we say that the field, Φ, has a non-zero vacuum expectation value

$$\langle\Phi\rangle \;=\; \frac{v}{\sqrt{2}}.$$

- We will see that this means that there are ‘excitations’ with zero energy

- The only particles which can have zero energy are massless particles (with zero momentum). We therefore expect a massless particle in such a theory.

- To see this, we expand Φ around its vacuum expectation value as

$$\Phi \;=\; \frac{1}{\sqrt{2}}\left(\frac{\mu}{\sqrt{\lambda}} + H + i\phi\right). \tag{5.5}$$

- The fields H and ϕ have zero vacuum expectation value and it is these fields that are expanded in terms of creation and annihilation operators of the particles that populate the excited states.

- Now insert (5.5) into (5.3) we find

$$V \;=\; \mu^2 H^2 + \mu\sqrt{\lambda}\left(H^3 + \phi^2 H\right) + \frac{\lambda}{4}\left(H^4 + \phi^4 + 2H^2\,\phi^2\right) + \frac{\mu^4}{4\lambda}. \tag{5.6}$$

- Note that in (5.6) there is a mass term for the field H, but *no* mass term for the field ϕ. Thus ϕ is a field for a massless particle called a "Goldstone boson".

- The field H will be "held" in a restoring potential, which corresponds precisely to a genuine mass term.

- On the other hand, in the ϕ direction at this point, the field will be moving along the "valley" of the potential which means that it's a massless field when quantized.

5.4 Goldstone Bosons

- Goldstones theorem extends this to spontaneous breaking of a general symmetry.
- Suppose we have a theory which is invariant under a symmetry group $\mathcal{G}$ with N generators
- Assume some operator (i.e. a function of the quantum fields - which might just be a component of one of these fields) has a non-zero vacuum expectation value, which breaks the symmetry down to a subgroup $\mathcal{H}$ of $\mathcal{G}$, with n generators
- The vacuum state is still invariant under transformations generated by the n generators of $\mathcal{H}$, but not the remaining $N-n$ generators of the original symmetry group $\mathcal{G}$.
- Thus we have
$$T^a|0\rangle = 0, \quad a = 1...n \qquad , \quad T^a|0\rangle \neq 0, \quad a = n+1...N$$
- Goldstone's theorem states that there will be $N-n$ massless particles (one for each broken generator of the group).
- The case considered in this section is special in, there is only one generator of the symmetry group (i.e. N = 1) which is broken by the vacuum.
- Thus, there is no generator that leaves the vacuum invariant (i.e. $n = 0$) and we get $N - n = 1$ Goldstone boson.
- The quantum numbers of the Goldstone particle are the same as the corresponding generator For example: Global transformations are usually internal transformations in field space not touching Lorentz indices
$\rightarrow$ Generators are Lorentz scalars - Goldstones are scalars
- Not true in SUSY; generators of SUSY transformations are fermionic - massless particles are fermions - Goldstinos.

5.5 The Higgs Mechanism

- Goldstone's theorem has a loophole, which arises when one considers a gauge theory, i.e. when one allows the original symmetry transformations to be local.
- In a spontaneously broken gauge theory, the choice of which vacuum is the true vacuum is equivalent to choosing a gauge, which is necessary in order to be able to quantise the theory.

- What this means is that the Goldstone bosons, which can, in principle, transform the vacuum into one of the states degenerate with the vacuum, now affect transitions into states which are not consistent with the original gauge choice.
- This means that the Goldstone bosons are "unphysical" and are often called "Goldstone ghosts".
- On the other hand the quantum degrees of freedom associated with the Goldstone bosons are certainly there before a choice of gauge is made. What happens to them?
- To see how this works we return to the $U(1)$ global theory, but now we promote the symmetry to a local symmetry (hence to a gauge theory)
- We must introduce a gauge boson, A_μ. The partial derivative of the field Φ is replaced by a covariant derivative

$$\partial_\mu \Phi \;\rightarrow\; D_\mu \Phi \;=\; (\partial_\mu + i\, g\, A_\mu)\, \Phi.$$

Including the kinetic term $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ for the gauge bosons, the Lagrangian density becomes

$$\mathcal{L} \;=\; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^* \, D^\mu \Phi - V(\Phi) \tag{5.7}$$

- Now note what happens if we insert the expansion

$$\Phi = \frac{1}{\sqrt{2}} \left(v + H + i\phi\right)$$

into the term $(D_\mu \Phi)^* \, D^\mu \Phi$. This generates the following terms

$$\begin{aligned}
(D_\mu \Phi)\,(D^\mu \Phi)^* \;&=\; \left[(\partial_\mu + igA_\mu)\left(\frac{1}{\sqrt{2}}(v+H+i\phi)\right)\right]\left[(\partial^\mu - igA^\mu)\left(\frac{1}{\sqrt{2}}(v+H-i\phi)\right)\right] \\
&=\; \left[\frac{1}{\sqrt{2}}\partial_\mu H + \frac{i}{\sqrt{2}}\partial_\mu \phi + \frac{ig}{\sqrt{2}} v A_\mu + \frac{ig}{\sqrt{2}} A_\mu H - \frac{g}{\sqrt{2}} A_\mu \phi\right] \\
&\times\; \left[\frac{1}{\sqrt{2}}\partial^\mu H - \frac{i}{\sqrt{2}}\partial^\mu \phi - \frac{ig}{\sqrt{2}} v A^\mu - \frac{ig}{\sqrt{2}} A^\mu H - \frac{g}{\sqrt{2}} A^\mu \phi\right] \\
&=\; \frac{1}{2}\partial_\mu H\, \partial^\mu H + \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi + \frac{1}{2} g^2 v^2 A_\mu A^\mu \\
&+\; \frac{1}{2} g^2 A_\mu A^\mu \left(H^2 + \phi^2\right) - \, g\, A^\mu \left(\phi\, \partial_\mu H \,-\, H\, \partial_\mu \phi\right) + g\, v\, A^\mu \partial_\mu \phi + g^2\, v\, A^\mu A_\mu H,
\end{aligned}$$

(where $v = \mu/\sqrt{\lambda}$). The gauge boson has acquired a mass term,

$$M_A \;=\; g\, v,$$

- There is a coupling of the gauge field to the H-field,

$$g^2 v A_\mu A^\mu H = g M_A A_\mu A^\mu H$$

- There is also the bilinear term

$$g v A_\mu \partial^\mu \phi$$

which after integrating by parts (for the action S) may be written as

$$-M_A \phi \partial_\mu A^\mu$$

- This mixes the Goldstone boson, ϕ, with the longitudinal component of the gauge boson, with strength M_A

- Explicitly,

$$M_A \phi \partial^\mu A_\mu = M_A \phi \partial^\mu (A^L_\mu + A^T_\mu) = M_A \phi \partial^\mu A^L_\mu$$

as $\partial^\mu A^T_\mu = 0$.

- Later on, we will use the gauge freedom to get rid of this mixing term.

- A massless vector boson has only two degrees of freedom (the two directions of polarisation of a photon)

- A massive vector (spin-one) particle has three possible values for the helicity of the particle, 2 transverse and 1 longitudinal

- In a spontaneously broken gauge theory, the Goldstone boson associated with each broken generator provides the third degree of freedom to the gauge bosons.
  $\rightarrow$ This means that the gauge bosons become massive.

- The Goldstone boson is said to be "eaten" by the gauge boson.

- This is related to the mixing term between A^L_μ and ϕ

- Thus, in our abelian model, the two degrees of freedom of the complex field Φ turn out to be the Higgs field and the longitudinal component of the (now massive) gauge boson.

- There is no physical, massless particle associated with the degree of freedom ϕ present in Φ.

5.6 Gauge Fixing

- Going back to the bilinear term

$$M_A \phi \partial^\mu A^L_\mu$$

 we can think of the longitudinal component of the gauge boson oscillating between the Goldstone boson due to this mixing term
 $\rightarrow$ the physical particle is described by a superposition of these fields.

We consider two special cases:

The unitary gauge:

- The physical field for the longitudinal component of the gauge boson is not simply A^L_μ, but the superposition

$$A^{ph}_\mu \; = \; A_\mu \; + \; \frac{1}{M_A}\partial_\mu \phi. \tag{5.8}$$

 (Note that this only affects the longitudinal component).

- Now all the terms quadratic in A_μ, ϕ (including the bilinear mixing term) may be written (in momentum space) as

$$A^{ph}_\mu(-p)\left(-g^{\mu\nu}\,p^2 \; + \; p^\mu p^\nu + g^{\mu\nu} M_A^2\right) A^{ph}_\nu(p). \tag{5.9}$$

- Writing out the Lagrangian we notice that the Goldstone boson field, ϕ, has *disappeared*. The terms involving ϕ in the original expression have been absorbed (or "eaten") by the redefinition (5.8) of the gauge field.

- The gauge boson propagator is the inverse of the coefficient of (dropping the superscript *ph*) $A_\mu(-p)A_\nu(p)$ in (5.9) , which is

$$-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M_A^2}\right)\frac{1}{(p^2 - M_A^2)}, \tag{5.10}$$

 which is the usual expression for the propagator of a massive spin-one particle.

- The only other remaining particle is the scalar, H, with mass $m_H \; = \; \sqrt{2}\,\mu$ which is called the "Higgs" particle

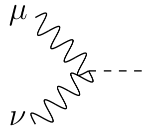
- This is a physical particle, which interacts with the gauge boson and also has cubic and quartic self-interactions.

- The interaction terms involving the Higgs boson are

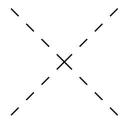
$$\mathcal{L}_I(H) \;=\; \frac{e^2}{2}A_\mu A^\mu H^2 + eM_A A_\mu A^\mu H - \frac{\lambda}{4}H^4 - m_H\sqrt{2\lambda}H^3, \tag{5.11}$$

which leads to the following vertices and Feynman rules.

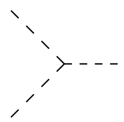
$2\,i\,e^2 g_{\mu\nu}$



$2\,i\,eM_A g_{\mu\nu}$



$6\,i\,\lambda$



$6\,i\,m_H\sqrt{2\lambda}$

- Advantage of the unitary gauge is that no unphysical particles appear, no ϕ fields appear
- The disadvantage is that the propagator of the gauge field behaves as a constant for $p \to \infty$.
- As we have discussed this seems to indicate that the theory is non-renormalizable.
- Fortunately this is not true. In order to see that the theory is still renormalizable it is very useful to consider a different type of gauges, namely the R_ξ gauge.

Another Quick look at the Unitary Gauge

- Now use "polar" field coordinates (radial and angle variables) rather than the "Cartesian" ones above. We set

$$\phi = \frac{1}{\sqrt{2}}(v + \rho(x))\; e^{i\theta(x)/v}.$$

- The "radial" field ρ and the "angle" field θ here replace the "Cartesian" H and ϕ.
- There are still, of course, two (field) degrees of freedom.

- Stick this into L and find

$$L = (\frac{1}{2}\partial_\mu\rho\ \partial^\mu\rho \quad - \frac{1}{2}\mu^2\rho^2) + (\frac{1}{2}\partial_\mu\theta\ \partial^\mu\theta) + \frac{\rho}{v}\partial_\mu\theta\ \partial^\mu\theta\ + \text{other interaction terms.}$$

- The "θ" mode is the one for motion around the equilibrium circle, and it is massless.

- The "ρ" mode (radial) is restored, and has mass μ.

- Now make the global $U(1)$ symmetry of this model into a local symmetry by introducing a $U(1)$ gauge field

- This is the **Abelian Higgs model**.

- We have to change all derivatives to covariant derivatives and add in the Maxwell term for the A_μ field. This produces

$$L = [(\partial_\mu + igA_\mu)\phi]^\dagger\ [(\partial^\mu + igA^\mu)\phi] \quad - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad + \quad \frac{1}{2}\mu^2 \mid \phi \mid^2 \quad - \quad \frac{1}{2}\lambda^2 \mid \phi \mid^4 .$$

- As in the Goldstone model, expand about a point on the equilibrium circle as before

- This time we shall choose to use the "polar" field variables, and write

$$\phi = \frac{1}{\sqrt{2}}(v + \rho(x))\ e^{i\theta(x)/v}$$

  where $v = \mu/\lambda$.

- The theory is invariant under the combined transformations

$$\phi \to \phi' = e^{-i\chi(x)}\ \phi$$

  and

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{g}\partial_\mu\chi,$$

  where χ is arbitrary.

- The fields ρ and θ transform by

$$\rho \to \rho$$

  and

$$\theta \to \theta - v\chi.$$

- So, we can choose the field χ to be θ/v and so θ vanishes!

- That is, *we can choose a gauge in which ϕ is real.*
- Remember that the gauge transformation affects the gauge field A_μ and the complex scalar field ϕ simultaneously.
- After the gauge transformation in which θ is reduced to zero, A_μ is changed to

$$A'_\mu = A_\mu + \frac{1}{gv}\partial_\mu\theta$$

and ϕ is

$$\phi' = \frac{1}{\sqrt{2}}(v+\rho).$$

- L in terms of these primed fields is then

$$L(\phi', A'_\mu) = (\frac{1}{2}\partial_\mu\rho\ \partial^\mu\rho \ -\ \frac{1}{2}\mu^2\rho^2) \ \ -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} \ \ +\ \ \frac{1}{2}g^2v^2A'_\mu A'^\mu \ \ + \text{interactions}.$$

- ρ has a genuine mass μ.
- But where is the massless mode θ?
- It has been "eaten" by the A_μ field!
- It is present in A'_μ via

$$A'_\mu = A_\mu + \frac{1}{gv}\partial_\mu\theta.$$

- *A'_μ is a massive spin-1 particle, of mass gv*!
- This is the Higgs mechanism, whereby the massless gauge field A_μ has become a massive spin-1 field A'_μ by "eating" the scalar field θ.
- $\theta = 0$ is physically appealing and is called the "unitary gauge".

R_ξ **Feynman gauge:**

- We select the Feynman gauge by adding to the Lagrangian density the term

$$\begin{aligned}\mathcal{L}_R &\equiv -\frac{1}{2(1-\xi)}\left(\partial\cdot A - (1-\xi)M_A\phi\right)^2 \\ &= -\frac{1}{2(1-\xi)}\partial_\mu A^\mu\partial_\nu A^\nu + M_A\phi\partial_\mu A^\mu - \frac{1-\xi}{2}M_A^2\phi^2\end{aligned}$$

- The cross-term $M_A\phi\,\partial_\mu A^\mu$ cancels the bilinear mixing term.
- Again, the special value $\xi = 0$ corresponds to the Feynman gauge.
- The quadratic terms in the gauge boson are

$$\frac{1}{2}A^\mu(-p)\left(-g_{\mu\nu}(p^2 - M_A^2) + p_\mu p_\nu - \frac{p_\mu p_\nu}{1-\xi}\right)A^\nu(p)$$

  leading to the propagator

$$\frac{-i}{(p^2 - M_A^2)}\left(g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2 - (1-\xi)M_A^2}\right)$$

- in the Feynman gauge the gauge boson propagator simplifies to

$$-i\,\frac{g_{\mu\nu}}{(p^2 - M_A^2)}, \tag{5.12}$$

  which is easy to handle.
- There is, however, a price to pay. The Goldstone boson is still present.
- It has acquired a mass, M_A, from the gauge fixing term, and it has interactions with the gauge boson, with the Higgs scalar and with itself.
- Furthermore, for the purposes of higher order corrections in non-abelian theories we again need to introduce Faddeev-Popov ghosts which in this case interact not only with the gauge boson, but also with the Higgs scalar and Goldstone boson.

5.7 Bit more on Renormalisability

- Recalling the form of the gauge boson propagator fir the Unitary gauge

$$-i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M_A^2}\right)\frac{1}{(p^2 - M_A^2)} \tag{5.13}$$

  we see that it does *not* decrease as $p \to \infty$
- This would normally lead to a violation of renormalisability, thereby rendering the Quantum Field Theory useless
- However, there is no contradiction between the apparent non-renormalizability of the theory in the unitary gauge and the manifest renormalizability in the R_ξ gauge.

- Physical quantities are gauge invariant, any physical quantity can be calculated in a gauge where renormalizability is manifest.
- The price we pay for this is that there are more particles and many more interactions, leading to a plethora of Feynman diagrams.
- Only work in such gauges if we want to compute higher order corrections.
- For the rest of these lectures we shall confine ourselves to tree-level calculations and work solely in the unitary gauge.
- Nevertheless, one cannot over-stress the fact that it is only when the gauge bosons acquire masses through the Higgs mechanism that we have a renormalizable theory.

5.8 Spontaneous Symmetry Breaking in a Non-Abelian Gauge Theory

- Extend to non-Abelian gauge theories.
- Take an $SU(2)$ gauge theory and consider a complex doublet of scalar fields. $\Phi^i, \; i = 1, 2.$
- The Lagrangian density is

$$\mathcal{L} \; = \; -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + |\mathbf{D}_\mu \Phi|^2 - V(\Phi), \tag{5.14}$$

where

$$\mathbf{D}_\mu \Phi \; = \; \partial_\mu \Phi + i\, g\, W^a_\mu \, \mathbf{T}^a \, \Phi,$$

(we have changed notation for the gauge bosons from A^a_μ to W^a_μ), and

$$V(\Phi) \; = \; -\mu^2 \Phi^\dagger_i \Phi^i + \lambda \left(\Phi^\dagger_i \Phi^i \right)^2 . \tag{5.15}$$

- This potential has a minimum at $\Phi^\dagger_i \Phi_i = \frac{1}{2} \mu^2 / \lambda$.
- We choose the vacuum expectation value to be in the $T^3 = -\frac{1}{2}$ direction and to be real, i.e.

$$\langle \Phi \rangle \; = \; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix},$$

$(v = \mu / \sqrt{\lambda})$.

- This vacuum expectation value is not invariant under *any* $SU(2)$ transformation.
- This means that there is *no* unbroken subgroup
- Counting goes as follows: We start with 2 component complex scalar = 4 dof
  three generators are broken = three Goldstone bosons with all three of the gauge bosons acquiring a mass
  one dof left which is the massive Higgs scalar.

- We expand Φ^i about its vacuum expectation value ("vev")

$$\Phi \; = \; \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 - i\, \phi_2 \\ v \, + \, H \, + \, i\, \phi_0 \end{pmatrix}.$$

- The $\phi_a, \; a = 0 \cdots 2$ are the three Goldstone bosons and H is the physical Higgs scalar.

- All of these fields have zero vev.

- If we insert this expansion into the potential (5.15) then we find that we only get a mass term for the Higgs field, with value $m_H = \sqrt{2}\mu$.

- For simplicity, move directly into the unitary gauge by setting all the three ϕ_a to zero.

- In this gauge $\mathbf{D}_\mu\Phi$ may be written

$$\begin{aligned}\mathbf{D}_\mu\Phi &= \frac{1}{\sqrt{2}}\left(\partial_\mu\begin{pmatrix}0\\ H\end{pmatrix} + i\,\frac{g}{2}\begin{pmatrix}W^3_\mu & W^1_\mu - iW^2_\mu\\ W^1_\mu + iW^2_\mu & -W^3_\mu\end{pmatrix}\begin{pmatrix}0\\ v+H\end{pmatrix}\right)\\ &\frac{1}{\sqrt{2}}\left(\partial_\mu\begin{pmatrix}0\\ H\end{pmatrix} + i\,\frac{g}{2}\begin{pmatrix}W^0_\mu & \sqrt{2}W^+_\mu\\ \sqrt{2}W^-_\mu & -W^0_\mu\end{pmatrix}\begin{pmatrix}0\\ v+H\end{pmatrix}\right),\end{aligned}$$

  where we have introduced the notation $W^\pm_\mu = (W^1_\mu \mp i\,W^2_\mu)/\sqrt{2}$, $W^0_\mu = W^3_\mu$ and used the explicit form for the generators of $SU(2)$ in the 2×2 representation given by the Pauli matrices.

- The term $|\mathbf{D}_\mu\Phi|^2$ then becomes

$$\begin{aligned}|\mathbf{D}_\mu\Phi|^2 &= \frac{1}{2}\partial_\mu H\,\partial^\mu H + \frac{1}{4}g^2v^2\left(W^+_\mu W^{-\,\mu} + \frac{1}{2}W^0_\mu W^{0\,\mu}\right)\\ &+\frac{1}{4}g^2H^2\left(W^+_\mu W^{-\,\mu} + \frac{1}{2}W^0_\mu W^{0\,\mu}\right) + \frac{1}{2}g^2vH\left(W^+_\mu W^{-\,\mu} + \frac{1}{2}W^0_\mu W^{0\,\mu}\right). \qquad (5.16)\end{aligned}$$

- We see from the terms quadratic in W_μ that all three of the gauge bosons have acquired a mass

$$M_W = \frac{gv}{2}$$

6 Spontaneous Breaking of $SU(2)_L \times U(1)_Y \to U(1)_{\rm EM}$

- Consider the particle content just consisting of one Higgs doublet, Φ and the gauge fields associated with the $SU(2)_L \times U(1)_Y$
- Higgs has quantum number $(\mathbf{2}, \frac{1}{2})$ and has the form

$$\Phi = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$$

- The Lagrangian for this is then

$$\mathcal{L} \;=\; -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + |\mathbf{D}_\mu\Phi|^2 - V(\Phi) \qquad \text{(4 d.o.f)}$$

where

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\, f^{abc}\, W^b_\mu W^c_\nu$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$\mathbf{D}_\mu\,\Phi = \partial_\mu + \frac{ig'}{2}B_\mu + \frac{ig}{2}\tau^i W^i_\mu$$
$$V(\Phi) \;=\; -\mu^2\Phi^\dagger_i\Phi^i + \lambda\left(\Phi^\dagger_i\Phi^i\right)^2$$

where τ^is are the Pauli matrices.

- Expanding the Higgs field around its VEV In the Unitary gauge the covariant derivative takes the form

$$\begin{aligned}
\mathbf{D}_\mu\,\Phi &= \frac{1}{\sqrt{2}}\left(\partial_\mu \;+ i\,\frac{g}{2}\begin{pmatrix} W^0_\mu & \sqrt{2}\,W^+_\mu \\ \sqrt{2}\,W^-_\mu & -W^0_\mu \end{pmatrix} + i\,\frac{g'}{2}B_\mu\right)\begin{pmatrix} 0 \\ v\;+\;H \end{pmatrix} \qquad (6.1)\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} \partial_\mu + i\,\frac{g}{2}W^0_\mu + i\,\frac{g'}{2}B_\mu & i\,\frac{g}{2}\sqrt{2}\,W^+_\mu \\ i\,\frac{g}{2}\sqrt{2}\,W^-_\mu & \partial_\mu - i\,\frac{g}{2}W^0_\mu + i\,\frac{g'}{2}B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v\;+\;H \end{pmatrix}\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} i\,\frac{g}{2}\sqrt{2}\,W^+_\mu(v+H) \\ (\partial_\mu - i\,\frac{g}{2}W^0_\mu + i\,\frac{g'}{2}B_\mu)(v+H) \end{pmatrix}
\end{aligned}$$

where we have taken $W^\pm_\mu = \frac{W^1_\mu \mp i W^2_\mu}{\sqrt{2}}$ and $W^0_\mu = W^3_\mu$. We then get

$$\begin{aligned}
|\mathbf{D}_\mu\,\Phi|^2 \;=\; & \frac{1}{2}(\partial_\mu\,H)^2 \;+\; \frac{g^2v^2}{4}\,W^{+\,\mu}W^-_\mu + \frac{v^2}{8}\left(g\,W^0_\mu - g'B_\mu\right)^2 \;+ \\
& \frac{g^2}{4}\,W^{+\,\mu}W^-_\mu(2vH + H^2) + \frac{1}{8}\left(g\,W^0_\mu - g'B_\mu\right)^2(2vH + H^2)\;. \qquad (6.2)
\end{aligned}$$

- We get a mass terms for the charged gauge boson and a linear combination $\left(g\, W_\mu^0 - g' B_\mu\right)$.

- We need to diagonalise the $W_\mu^0 - B_\mu^0$ system and we can do that by introducing

$$\begin{aligned} Z_\mu &= \cos\theta_w W_\mu^0 - \sin\theta_w B_\mu \\ A_\mu &= \sin\theta_w W_\mu^0 + \cos\theta_w B_\mu \end{aligned}$$

i.e.

$$\begin{aligned} B_\mu &= \cos\theta_w A_\mu - \sin\theta_w Z_\mu \\ W_\mu^0 &= \sin\theta_w A_\mu + \cos\theta_w Z_\mu \end{aligned}$$

where

$$\tan\theta_w = \frac{g'}{g}$$

- Now our non-interaction part of the Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{\mathrm{non-int}} &= \frac{1}{2}\left((\partial.H)^2 - \mu^2 H^2\right) \\ &- \frac{1}{4}\left(\partial_\mu W_\nu^1 - \partial_\nu W_\mu^1\right)\left(\partial^\mu W^{\nu 1} - \partial^\nu W^{\mu 1}\right) + \frac{1}{8} g^2 v^2 W_1^2 \\ &- \frac{1}{4}\left(\partial_\mu W_\nu^2 - \partial_\nu W_\mu^2\right)\left(\partial^\mu W^{\nu 2} - \partial^\nu W^{\mu 2}\right) + \frac{1}{8} g^2 v^2 W_2^2 \\ &- \frac{1}{4}\left(\partial_\mu Z_\nu - \partial_\nu Z_\mu\right)\left(\partial^\mu Z^\nu - \partial^\nu Z^\mu\right) + \frac{1}{8}(g^2 + g'^2) v^2 Z^2 \\ &- \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \end{aligned}$$

- The physical interpretation of this is the following: 3 massive vector particles

$$M_1 = M_2 = \frac{gv}{2} = M_{W^\pm}$$

and

$$\frac{v}{2}(g^2 + g'^2)^{1/2} = \frac{vg}{2}\frac{1}{\cos\theta_w} = \frac{M_{W^\pm}}{\cos\theta_w} = M_Z$$

- We also have one massless vector and a massive Higgs

- The masslessness of A_μ follows from the choice of breaking condition, i.e. the form of the vev.

- There is a combination of the original generators that still annihilates the the vacuum, i.e. there is a linear combinations of generators that is still invariant - one gauge symmetry not spontaneously broken

$$\begin{aligned}\left(Y+\frac{T_3}{2}\right)\begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix} &= \left[\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}+\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}\right]\begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\end{aligned}$$

- We can rewrite this linear combination in terms of the isospin $\tau_3 = 2t_3^{\text{isospin}}$ so that the linear combination is

$$Y + t_3^{\text{isospin}}$$

- Going back to the covariant derivative for the doublet Φ we have

$$\begin{aligned}\mathbf{D}_\mu \Phi &= \partial_\mu\Phi + \frac{ig'}{2}B_\mu\Phi + \frac{ig}{2}\tau^i W^i_\mu\Phi\\ &= \partial_\mu\Phi + \frac{ig'}{2}\left(\cos\theta_w A_\mu - \sin\theta_w Z_\mu\right)\Phi + \frac{ig}{2}(\tau^3\sin\theta_w A_\mu + \cos\theta_w Z_\mu)\Phi + W^1,\ W^2 \text{ bits}\\ &= \partial_\mu\Phi + \frac{ig}{2}\tan\theta_w\left(\cos\theta_w A_\mu - \sin\theta_w Z_\mu\right)\Phi + \frac{ig}{2}(\tau^3\sin\theta_w A_\mu + \cos\theta_w Z_\mu)\Phi\\ &= \partial_\mu\Phi + \frac{ig}{2}\sin\theta_w(1+\tau^3)A_\mu\Phi + \frac{ig}{\cos\theta_w}\left(\frac{\tau^3}{2}-\sin^2\theta_w\frac{(1+\tau^3)}{2}\right)Z_\mu\Phi\\ &\Rightarrow\ \mathbf{D}_\mu = \partial_\mu + ig\sin\theta_w\left(\frac{1}{2}+t_3^{\text{isospin}}\right)A_\mu + \frac{ig}{\cos\theta_w}\left(\frac{\tau^3}{2}-\sin^2\theta_w\frac{(1+\tau^3)}{2}\right)Z_\mu\end{aligned}$$

We then of course identify

$$g\sin\theta_w\left(\frac{1}{2}+t_3^{\text{isospin}}\right) = eQ(\Phi)$$

with

$$g\sin\theta_w = e$$

and

$$Q(\Phi) = \frac{1}{2}+t_3^{\text{isospin}}$$

- We can see this makes sense as

$$\begin{aligned}g\sin\theta_w\left(\frac{1}{2}+t_3^{\text{isospin}}\right)\Phi &= g\sin\theta_w\left(\frac{1}{2}+t_3^{\text{isospin}}\right)\begin{pmatrix}h^+\\ h^0\end{pmatrix}\\ &= g\sin\theta_w\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}h^+\\ h^0\end{pmatrix} = e\begin{pmatrix}Q & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}h^+\\ h^0\end{pmatrix} \qquad (6.3)\end{aligned}$$

if we are to identify A_μ as the photon

6.1 More on renormalisability

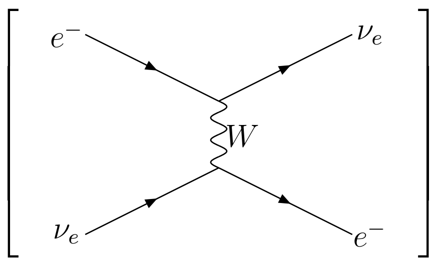
Electroweak theory is renormalisable

- Hopefully the LHC will determine whether the simplest Higgs model is correct
- Higgs is motivated by theory - no experimental evidence - or for any fundamental scalar.
- Allows us to introduce weak boson masses without spoiling renormalisability of electroweak theory
- We have seen a rough argument with propagators but we can also see the effects in diagrams.
- Structure of lowest-order amplitudes for weak processes hints that we may have trouble without the Higgs.
- Consider the cross section

$$\sigma(\nu_e e \to \nu_e e) = \frac{G^2 s}{\pi}$$

- Becomes infinite as $s \to \infty$.
- This cannot be true for large s as this violates unitarity.
- Can solve this by introducing a W mass and now for large s we have a diagram

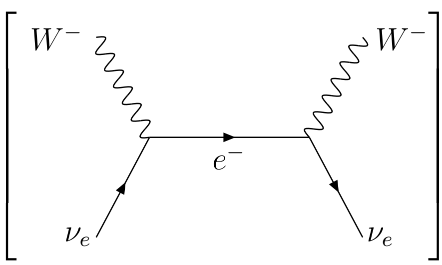
σ



$$= \frac{G^2 M_W^2}{\pi}$$

- **However** the introduction of the W boson causes problems due to the diagram

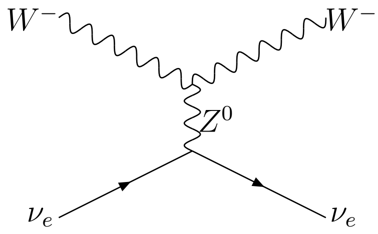
σ



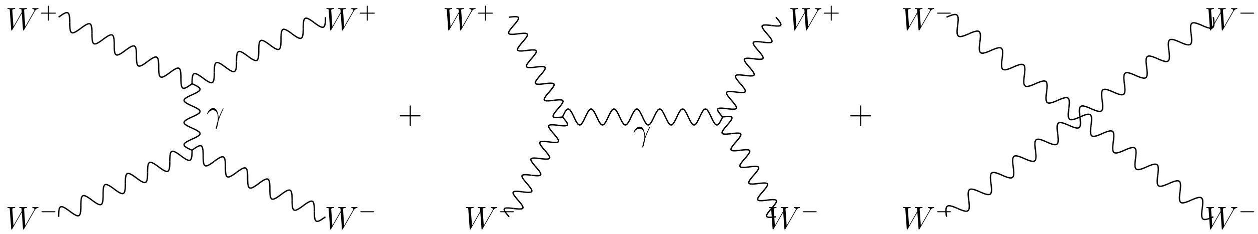
$$= \frac{G^2 s}{3\pi}$$

which diverges at large s.

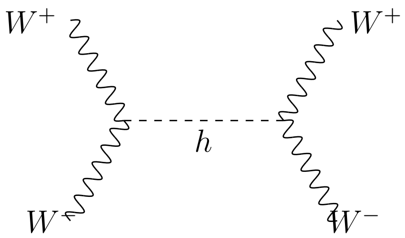
- However we also have the diagram proceeding via neutral current



- this diagram actually cancels the divergence from the charged current diagram
- Beautiful demonstration of gauge theory - gauge boson self-interactions ensuring a finite answer
- Another example...WW scattering



- We also have diagrams via Z exchange, when we collect these all together we find that the result diverges when $s \to \infty$
- Only way we can cure this is by introducing a scalar particle which cancels these divergences through a similar diagram



- Of course h is the Higgs boson.
- If we had not already introduced the Higgs to generate heavy boson masses we would have been forced to introduce it to guarantee renormalisability - i.e. to unitarise the WW scattering cross section
- Analysing this further shows that the Higgs couplings must be proportional to the particle masses in order to cancel the divergences - more evidence in favour of the structure of the SM

6.2 Summary

- For a field theory in which the potential is not a minimum when all the fields take the value zero, at least one of the fields acquires a non-zero vacuum expectation value (vev).
- The symmetry is said to be spontaneously broken because one of many degenerate ground states is chosen to be the true vacuum.
- There may, in general, be a subset of transformations (i.e. a subgroup $\mathcal{H}$ of the original symmetry group $\mathcal{G}$) under which the vev is invariant.
- The vacuum is then invariant under this subgroup $\mathcal{H}$ and we say that the original symmetry described by the group $\mathcal{G}$ has been spontaneously broken to the subgroup $\mathcal{H}$.
- When a field, in a theory that is invariant under a set of transformations, acquires a vacuum expectation value there is a massless Goldstone boson for each generator of the symmetry, which is broken by that expectation value.
- These Goldstone bosons are the 'excitations' which effect transitions to the other states, which are degenerate with the vacuum.
- In the case of a gauge theory these Goldstone bosons provide the longitudinal component of the gauge bosons, which therefore acquire a mass.
- The mass is proportional to the magnitude of the vacuum expectation value and the gauge coupling constant.
- The Goldstone bosons themselves are unphysical.
- One can work in the unitary gauge where the Goldstone boson fields are set to zero.
- When gauge bosons acquire masses by this (Higgs) mechanism, renormalisability is maintained.
- This can be seen explicitly if one works in a gauge other than the unitary gauge, in which the gauge boson propagator decreases as $1/p^2$ as $p \to \infty$, which is a necessary condition for renormalisability.
- If one does work in such a gauge, however, one needs to work with Goldstone boson fields, even though the Goldstone bosons are unphysical.
- The number of interactions and the number of Feynman graphs required for the calculation of some process is greatly increased.

7 The Electroweak Model of Leptons

- Only one or two modifications are needed to the model described at the end of the last lecture to obtain the Glashow-Weinberg-Salam (GWS) model of electroweak interactions.
- This was the first model that successfully unified different forces of Nature.
- In this lecture we shall consider only leptons as matter fields, deferring the introduction of hadrons to the next lecture.

7.1 Left- and right- handed fermions

- The weak interactions are known to violate parity
  $\rightarrow$ they are not symmetric under interchange of left-helicity and right-helicity fermions.

First Recall a few things about writing Lagrangians in terms of Weyl and Dirac spinors.

- A Dirac field, ψ, representing a fermion, can be expressed as the sum of a left-handed part, ψ_L, and a right-handed part, ψ_R,

$$\psi = \psi_L + \psi_R, \tag{7.1}$$

where

$$\begin{aligned} \psi_L &= P_L \psi \\ P_L &= \frac{(1-\gamma_5)}{2} \end{aligned} \tag{7.2}$$

$$\begin{aligned} \psi_R &= P_R \psi \\ P_R &= \frac{(1+\gamma_5)}{2} \end{aligned} \tag{7.3}$$

P_L and P_R are projection operators in the sense that

$$P_L P_L = P_L, \quad P_R P_R = P_R, \quad \text{and } P_L P_R = 0. \tag{7.4}$$

- They project out the left-handed (negative) and right-handed (positive) helicity states of the fermion respectively.

- The kinetic term of the Dirac Lagrangian and the interaction term of a fermion with a vector field can also be written as a sum of two terms each involving only one helicity.

$$\overline{\psi}\,\gamma^\mu\partial_\mu\psi \;=\; \overline{\psi_L}\,\gamma^\mu\partial_\mu\psi_L \;+\overline{\psi_R}\,\gamma^\mu\partial_\mu\psi_R, \tag{7.5}$$

$$\overline{\psi}\,\gamma^\mu A_\mu\psi \;=\; \overline{\psi_L}\,\gamma^\mu A_\mu\psi_L \;+\overline{\psi_R}\,\gamma^\mu A_\mu\psi_R. \tag{7.6}$$

- On the other hand, a mass term mixes the two helicities

$$m\overline{\psi}\,\psi \;=\; m\overline{\psi_L}\,\psi_R \;+\; m\overline{\psi_R}\,\psi_L.$$

- Thus, if the fermions are massless, we can treat the left-handed and right-handed helicities as separate particles.

- We can understand this physically:
  Massive fermion moving along the *positive* z-axis with spin *positive* component, so that the helicity is *positive* in this frame

- Can always boost into a frame with the fermion moving along the *negative* z-axis, but the component of spin is unchanged, helicity will be *negative*.

- For massless particle it travels with the speed of light no such boost is possible and in that case the helicity is a good quantum number.

- Since charged weak interactions are observed to occur only between left-helicity fermions, we consider a weak isospin, $SU(2)$, under which the left-handed leptons transform as a doublet

$$L_L \;=\; \begin{pmatrix} \nu_L \\ e_L \end{pmatrix},$$

  under this $SU(2)$, but the right-handed electron e_R is a singlet.

- i.e

$$e_R \to e'_R = e_R$$

  where as

$$L_L \to L'_L = e^{-\omega^a \mathbf{T^a}} L_L$$

- Under $U(1)_Y$ gauge transformations

$$e_R \to e'_R = e^{-\omega Y(e_R)} e_R$$

where as

$$L_L \to L'_L = e^{-\omega Y(L_L)} L_L$$

with

$$Y(e_R) = -1 \qquad \text{and} \qquad Y(L_L) = -\frac{1}{2}$$

- Since this separation of the electron into its left- and right-handed helicity only makes sense for a massless electron we also need to assume that the electron *is* massless in the exact $SU(2)$ limit
- Also that the mass for the electron arises as a result of spontaneous symmetry breaking in the same way that the masses for the gauge bosons arise.

7.2 Fermion masses - Yukawa couplings

- Cannot have explicit mass terms for the electrons since a mass term mixes left-handed and right-handed fermions
- Is not gauge invariant.
- Can have an interaction between the left-handed lepton doublet, the right-handed electron and the scalar doublet Φ.
- Such an interaction is called a "Yukawa interaction" and is written

$$\mathcal{L}_{Yukawa} \; = \; -\lambda_e \, \overline{l_L}^i \, \Phi_i e_R \; + \; \text{h.c.} \tag{7.7}$$

Note that this term has zero weak hypercharge. In the unitary gauge this is

$$-\frac{\lambda_e}{\sqrt{2}} \begin{pmatrix} \overline{\nu_L} & \overline{e_L} \end{pmatrix} \begin{pmatrix} 0 \\ v \; + \; H \end{pmatrix} e_R \quad + \; \text{h.c.}.$$

The part proportional to the vev is simply

$$-\frac{\lambda_e \, v}{\sqrt{2}} \left(\overline{e_L} \, e_R + \overline{e_R} \, e_L\right) \; = -\frac{\lambda_e \, v}{\sqrt{2}} \left(\overline{e} P_R \, e + \overline{e} P_L \, e\right) \; = \; \frac{\lambda_e \, v}{\sqrt{2}} \, \overline{e} \, e,$$

and we see that the electron has acquired a mass, which is proportional to the vev of the scalar field.

- This immediately gives us a relation for the Yukawa coupling in terms of the electron mass, m_e, and the $W-$mass, M_W,

$$\lambda_e \; = \; g \, \frac{m_e}{\sqrt{2} \, M_W}.$$

- There is, moreover, a Yukawa coupling between the electron and the scalar Higgs field

$$- g \frac{m_e}{2 M_W} \overline{e} H e.$$

- Note that there is *no* coupling between the neutrino and the Higgs, (and of course no neutrino mass term).

7.3 Weak Interactions of Leptons

- The fermionic part of the Lagrangian is

$$\mathcal{L}_{Fermi} = i \overline{l_L} \gamma^\mu \mathbf{D}_\mu l_L + i\overline{e_R} \gamma^\mu D_\mu e_R, \tag{7.8}$$

where the covariant derivatives are:

$$\begin{aligned} \mathbf{D}_\mu &= \partial_\mu - \frac{ig \tan\theta_w}{2} B_\mu + \frac{ig}{2} \tau^i W_\mu^i \\ D_\mu &= \partial_\mu - ig \tan\theta_w B_\mu \end{aligned} \tag{7.9}$$

- This gives the following interaction terms between the leptons and the gauge bosons:

$$-\frac{g}{2} \begin{pmatrix} \overline{\nu_L} & \overline{e_L} \end{pmatrix} \gamma^\mu \left(\begin{pmatrix} W_\mu^0 & \sqrt{2} W_\mu^- \\ \sqrt{2} W_\mu^+ & -W_\mu^0 \end{pmatrix} - \tan\theta_W B_\mu \right) \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} - i g \tan\theta_W \overline{e_R} \gamma^\mu B_\mu e_R.$$

- Expanding this out, using the physical particles Z_μ and A_μ in place of B_μ and W_μ^0 and using the projection operators for left- and right-handed fermions to write the terms in terms of 4-cpt Dirac spinors we obtain the following interactions:

1. A coupling of the charged vector bosons $W^\pm$, which mediate transitions between neutrinos and electrons with an interaction term

$$-\frac{g}{2\sqrt{2}} \overline{\nu} \gamma^\mu \left(1 - \gamma^5\right) e W_\mu^- + \text{h.c.}.$$

2. The usual coupling of the electron with the photon,

$$g \sin\theta_W \overline{e} \gamma^\mu e A_\mu.$$

3. The coupling of neutrinos to the neutral weak gauge boson, Z_μ,

$$-\frac{g}{4\cos\theta_W} \overline{\nu} \gamma^\mu \left(1 - \gamma^5\right) \nu Z_\mu.$$

4. The coupling of both the left- and right-handed electron to the Z

$$\frac{g}{4\cos\theta_W}\,\overline{e}\,\left(\gamma^\mu\left(1-\gamma^5\right)-4\sin^2\theta_W\gamma^\mu\right)\,e\,Z_\mu.$$

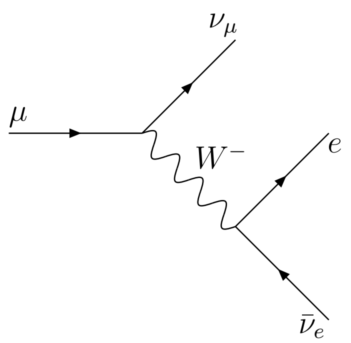
5. We can write this in a useful general form for later

$$\frac{g}{2\cos\theta_W}\,\overline{\psi_D}\,\left(\tau_3^{\text{isospin}}\gamma^\mu\left(1-\gamma^5\right)-2Q_i\sin^2\theta_W\gamma^\mu\right)\,\psi_D\,Z_\mu.$$

- We can include other lepton families, the muon and its neutrino, and the tau-lepton with its neutrino, as copies of what we have for the electron and its neutrino.

- For each family we have a weak isodoublet of left-handed leptons and a right-handed isosinglet for the charged lepton.

- The mechanism which determines the decay of the muon (μ) is one in which the muon converts into its neutrino and emits a charged W^-, which then decays into an electron and (anti-)neutrino. The Feynman diagram is



- The amplitude is given by the product of the vertex rules for the emission (or absorption) of a W^- with a propagator for the $W-$boson between them.

- Up to corrections of order m_μ^2/M_W^2, we may neglect the effect of the term $q^\mu q^\nu/M_W^2$ in the $W-$boson propagator, so that we have

$$\left(-\,i\,\frac{g}{2\sqrt{2}}\,\overline{\nu_\mu}\,\gamma^\rho(1-\gamma^5)\mu\right)\,\left(\frac{-\,i\,g_{\rho\sigma}}{q^2-M_W^2}\right)\,\left(-\,i\,\frac{g}{2\sqrt{2}}\,\overline{e}\,\gamma^\sigma(1-\gamma^5)\,\nu_e\right),$$

where q is the momentum transferred from the muon to its neutrino.

- Since this is negligible in comparison with M_W we may neglect it and the expression for the amplitude simplifies to

$$i\,\frac{g^2}{8M_W^2}\,\overline{\nu_\mu}\,\gamma^\rho(1-\gamma^5)\mu\,\overline{e}\,\gamma_\rho(1-\gamma^5)\nu_e.$$

- Before the development of this model, weak interactions were described by the "four-fermi model" with a weak interaction Hamiltonian given by

$$\mathcal{H}_{ijkl} = \frac{G_F}{\sqrt{2}} \overline{\psi}_i \gamma^\mu (1-\gamma^5)\psi_j \overline{\psi}_k \gamma_\mu (1-\gamma^5)\psi_l .$$

- This is an effective low-energy theory which may be used when the energy scales $q \ll M_W$.

- The Fermi coupling constant, G_F is related to the electric charge, e, the $W-$mass and the weak mixing angle by

$$G_F = \frac{e^2}{4\sqrt{2} M_W^2 \sin^2\theta_W}. \tag{7.10}$$

- This gives us a value for G_F,

$$G_F = 1.12 \times 10^{-5} \text{ GeV}^{-2}$$

which is very close to the value of 1.17×10^{-5} GeV^{-2} measured from the lifetime of the muon.

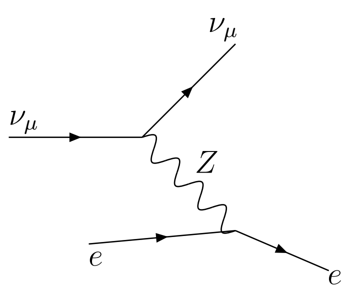
- Weak interactions are 'weak', not because the coupling is particularly small (the $SU(2)$ gauge coupling is about twice as large as the electromagnetic coupling), but because the exchanged boson is very massive, so that the Fermi coupling constant of the four-fermi theory is very small.

- The large mass of the $W-$boson is also responsible for the fact that the weak interactions are short range (of order 10^{-18} m).

- In SM we also have neutral weak currents.

- For example, we can have elastic scattering of muon-type neutrinos against electrons via the exchange of the $Z-$boson.

- The Feynman diagram for such a process is



7.4 Classifying the Free Parameters

The free parameters in the GWS model for one generation of leptons are

- The two gauge couplings for the $SU(2)$ and $U(1)$ gauge groups, g and g'.
- The two parameters, μ and λ in the scalar potential $V(\Phi)$
- The Yukawa coupling constant, G_e.

- It is convenient to replace these five parameters by five parameters which are directly measurable experimentally, namely e, $\sin\theta_W$, and three masses m_H, M_W, m_e.
- The relation between these "physical" parameters and the parameters that one writes down in the initial Lagrangian are

$$\begin{aligned}
\tan\theta_W &= \frac{g'}{g}, \\
e &= g\sin\theta_W, \\
m_H &= \sqrt{2}\mu, \\
M_W &= \frac{g\,\mu}{2\sqrt{\lambda}}, \\
m_e &= G_e\frac{\mu}{\sqrt{\lambda}}.
\end{aligned}$$

- Note that when we add more generations of leptons we acquire one more parameter for each added generation, namely the Yukawa coupling, which determines the mass of the charged lepton.
- In terms of these measured quantities, the $Z-$mass, M_Z, and the Fermi-coupling, G_F are *predictions* (although historically G_F was known for many years before the discovery of the $W-$boson and its value was used to predict the $W-$mass).

7.5 Summary

- Weak interactions are mediated by the gauge bosons of weak isospin which acts only on the left-handed helicity components of fermions.
- The neutrino and left-handed component of the electron form a doublet of this weak isospin whereas the right-handed component of the electron is a singlet.

- There is also a weak hypercharge $U(1)$ gauge symmetry. Both left- and right-handed leptons transform under this $U(1)$ with a hypercharge which is related to the electric charge of the electron by the relation

$$Q = Y + t_3^{\text{isospin}}$$

- The scalar multiplet which is responsible for the spontaneous symmetry breaking also carries weak hypercharge.

- As a result of this neutral gauge bosons mix in such a way that the superposition

$$\cos\theta_w W_\mu^0 - \sin\theta_w B_\mu$$

acquires a mass whereas its orthogonal superposition

$$\sin\theta_w W_\mu^0 + \cos\theta_w B_\mu$$

remains massless.

- The massive particle is the $Z-$boson and the massless particle is the photon.

- The magnitude of the electron charge, e, is then given by $e = g \sin\theta_W$.

- The weak interactions proceed via the exchange of massive charged or neutral gauge bosons.

- The old four-fermi weak Hamiltonian is an effective Hamiltonian which is valid for low energy processes in which all momenta are small compared with the $W-$mass.

- The Fermi coupling is obtained in terms of e, M_W and $\sin\theta_W$ by the relation

$$G_F = \frac{e^2}{4\sqrt{2}\, M_W^2 \, \sin^2\theta_W}$$

.

- In the symmetry limit the leptons are massless.

- The spontaneous symmetry breaking mechanism which gives a vev to the scalar field, also generates a mass for the electron.

- A second generation (μ, ν_μ) and third generation (τ, ν_τ) of leptons are added to the model as copies, with the only difference being that the Yukawa coupling is different for each generation, in order to generate the appropriate masses for the muon and tau-lepton respectively.

8 Electroweak Interactions of Hadrons

8.1 One Generation of Quarks

- We incorporate hadrons into the GWS model of electroweak interactions by adding weak isodoublets of left-handed quarks and weak isosinglets of right-handed quarks.
- A single generation consists of a $u-$quark and a $d-$quark.
- Each of these is a triplet of colour $SU(3)$ - we suppress this colour index for the moment but we must bear in mind that as far as the electroweak interactions are concerned we are really adding three copies of each of these quarks.
- Thus we have an isodoublet

$$q_L \;=\; \begin{pmatrix} u \\ d \end{pmatrix}_L$$

  and two isosinglets, u_R and d_R.
- These two quarks have electric charges of $+\frac{2}{3}$ and $-\frac{1}{3}$ respectively, so we adjust the weak hypercharge, Y, accordingly.
- The isodoublet, q_L has $Y = \frac{1}{6}$, whereas u_R and d_R have $Y = +\frac{2}{3}$ and $Y = -\frac{1}{3}$ respectively
- *no* reason why the right-handed components of the $u-$ and $d-$ quarks should have the same weak hypercharge, although the left-handed components *must* have the same weak hypercharge since they transform into each other under weak isospin transformations).
- The covariant derivatives are therefore given by:

$$\mathbf{D}_\mu q_L \;=\; \left(\partial_\mu \;+\frac{i\,g\,W_\mu^a \tau_i}{2} \;+\; i\,\frac{1}{6}\,g\,\tan\theta_W\;B_\mu\right) q_L, \tag{8.1}$$

$$D_\mu\,u_R = \left(\partial_\mu \;+\; i\,\frac{2}{3} g\,\tan\theta_W\;B_\mu\right) u_R, \tag{8.2}$$

$$D_\mu\,d_R = \left(\partial_\mu \;-\; i\,\frac{1}{3} g\,\tan\theta_W\;B_\mu\right) d_R. \tag{8.3}$$

These lead to the following interactions with the gauge bosons:

1. The coupling of the charged vector bosons $W^{\pm}$, which mediate transitions between $u-$ and $d-$ quarks are analogous to the lepton case. The interaction term is

$$-\frac{g}{2\sqrt{2}}\,\overline{u}\,\gamma^{\mu}\left(1-\gamma^{5}\right)d\,W_{\mu}^{-}\;+\text{h.c.}\;\;.$$

2. The couplings of the quarks with the photon give us the required quark charges,

$$-\frac{2}{3}\,g\,\sin\theta_W\,\overline{u}\,\gamma^{\mu}\,u\,A_{\mu}\;+\;\frac{1}{3}\,g\,\sin\theta_W\,\overline{d}\,\gamma^{\mu}\,d\,A_{\mu}.$$

3. The coupling of the quarks to the Z can be written in the general form

$$-\frac{g}{\cos\theta_W}\,\overline{q_i}\,\left(t_i^3\,\gamma^{\mu}\left(1-\gamma^{5}\right)-\,Q_i\sin^2\theta_W\gamma^{\mu}\right)\,q_i\,Z_{\mu},$$

where quark i has third component of weak isospin $t_i^3=(\pm 1/2,0)$ and electric charge Q_i.

- Once again, we can use these vertices to calculate weak interactions at the quark level.
- We can calculate the total decay width of the $Z-$ or $W-$ boson, by calculating the decay width into all possible quarks and leptons.
- However, for inclusive processes, in which we trigger on known initial or final state hadrons, information is needed about the probability to find a quark with given properties inside an initial hadron or the probability that a quark with given properties will decay ("fragment") into a final state hadron.

8.2 Quark Masses

- The quarks are also assumed to be massless in the symmetry limit and acquire a mass through the spontaneous symmetry breaking mechanism, via their Yukawa coupling with the scalars.
- The interaction term

$$-\,\lambda_d\,\overline{q_L}^{\,i}\,\Phi_i\,d_R\;+\;\text{h.c.}$$

gives a mass, m_d to the $d-$quark, when we replace Φ_i by its vev. This mass is given by

$$m_d\;=\;\frac{\lambda_d}{\sqrt{2}}\,v\;=\;\sqrt{2}\,\frac{\lambda_d\,M_W}{g}$$

- However, this does *not* generate a mass for the $u-$quark.

- In the case of $SU(2)$ there is a second way in which we can construct an invariant from such a Yukawa interaction.

- This is through the term

$$-\lambda_u\,\epsilon_{ij}\,\overline{q_L}^{\,i}\,\Phi^{\dagger\,j}u_R + \text{ h.c.},\quad (i,j\,=\,1,2), \tag{8.4}$$

where ϵ_{ij} is the two-dimensional antisymmetric tensor.

- Note that this term also has zero weak hypercharge as required by the $U(1)$ symmetry.

- This term does indeed give a mass m_u to the $u-$quark, where

$$m_u \;=\; \frac{\lambda_u}{\sqrt{2}}v \;=\; \sqrt{2}\,\frac{\lambda_u\,M_W}{g}$$

- The Higgs scalar couples to both the $u-$quark and the $d-$quark, with interaction terms

$$-\,g\frac{m_u}{2\,M_W}\,\overline{u}\,H\,u \;-\; g\frac{m_d}{2\,M_W}\,\overline{d}\,H\,d.$$

8.3 Adding Another Generation

- The second generation of quarks consists of a $c-$quark, which has electric charge $+\frac{2}{3}$ and an $s-$quark, with electric charge $-\frac{1}{3}$.

- We can just add a copy of the left-handed isodoublet and copies of the right-handed singlets in order to include this generation.

- The only difference would be in the Yukawa interaction terms where the coupling constants are chosen to reproduce the correct masses for the new quarks.

- In this case there is a further complication.

- It is possible to write down Yukawa terms which mix quarks of different generations, e.g.

$$\begin{aligned}-\,\lambda_{ds}\left(\begin{array}{cc}\overline{u_L} & \overline{d_L}\end{array}\right)\Phi\,s_R - \lambda_{dd}\left(\begin{array}{cc}\overline{u_L} & \overline{d_L}\end{array}\right)\Phi\,d_R - \lambda_{ss}\left(\begin{array}{cc}\overline{c_L} & \overline{s_L}\end{array}\right)\Phi\,s_R \\ = -\frac{\lambda_{ds}v}{\sqrt{2}}\overline{d_L}\;s_R - \frac{\lambda_{dd}v}{\sqrt{2}}\overline{d_L}\;d_R - \frac{\lambda_{ss}v}{\sqrt{2}}\overline{s_L}\;s_R\end{aligned}$$

- This term gives rise to a mass mixing between the $d-$quark and $s-$quark.

- The 'physical' particles are those that diagonalize the mass matrix.

- This means that the quarks which couple to the gauge bosons *are*, in general, superposition of physical quarks.

- Thus we write the two isodoublets of left-handed quarks as

$$\begin{pmatrix} u \\ \tilde{d} \end{pmatrix}_L ,$$

and

$$\begin{pmatrix} c \\ \tilde{s} \end{pmatrix}_L ,$$

where $\tilde{d}$ and $\tilde{s}$ are related to the physical $d-$quark and $s-$quark by

$$\begin{pmatrix} \tilde{d} \\ \tilde{s} \end{pmatrix} = \mathbf{V_C} \begin{pmatrix} d \\ s \end{pmatrix}, \tag{8.5}$$

where $\mathbf{V_C}$ is a unitary 2×2 matrix.

- Terms which are diagonal in the quarks are unaffected by this unitary transformation of the quarks.

- Thus the coupling to photons or $Z-$bosons is the same whether written in terms of $\tilde{d},\ \tilde{s}$ or simply $s,\ d$.

- On the other hand the coupling to the charged gauge bosons is

$$-\frac{g}{2\sqrt{2}}\,\overline{u}\,\gamma^\mu(1-\gamma^5)\,\tilde{d}\,W^-_\mu \;\; -\frac{g}{2\sqrt{2}}\,\overline{c}\,\gamma^\mu(1-\gamma^5)\,\tilde{s}\,W^-_\mu \quad + \text{h.c.},$$

which we may write as

$$-\frac{g}{2\sqrt{2}} \begin{pmatrix} \overline{u} & \overline{c} \end{pmatrix} \gamma^\mu(1-\gamma^5)\,\mathbf{V_C} \begin{pmatrix} d \\ s \end{pmatrix} W^-_\mu \quad + \text{h.c.}.$$

- The most general 2×2 unitary matrix may be written as

$$\begin{pmatrix} e^{-i\gamma} & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} e^{i\alpha} & \\ & e^{i\beta} \end{pmatrix}.$$

- We have set one of the phases to unity since we can always absorb an overall phase by adjusting the remaining phases, α, β, and γ.

- The phases, $\alpha,\ \beta,\ \gamma$ can be absorbed by performing a global phase transformation on the $d-$, $s-$ and $u-$quarks respectively.

- This again has no effect on the neutral terms.

- Thus the most general observable unitary matrix is given by

$$\mathbf{V_C} \ = \ \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}, \tag{8.6}$$

  where θ_C is the Cabibbo angle.

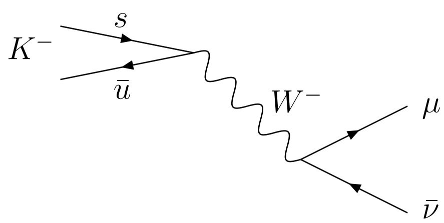
- In terms of the physical quarks (mass eigenstates), the charged gauge boson interaction terms are

$$-\frac{g}{2\sqrt{2}}\left(\cos\theta_C\,\overline{u}\,\gamma^\mu(1-\gamma^5)\,d \ + \ \sin\theta_C\,\overline{u}\,\gamma^\mu(1-\gamma^5)\,s \right.$$
$$\left. + \ \cos\theta_C\,\overline{c}\,\gamma^\mu(1-\gamma^5)\,s \ - \ \sin\theta_C\,\overline{c}\,\gamma^\mu(1-\gamma^5)\,d\right)\,W^-_\mu \quad + \text{h.c.} \tag{8.7}$$

- This means that the $u-$ quark can undergo weak interactions in which it is converted into an $s-$quark, with an amplitude that is proportional to $\sin\theta_C$.

- It is this that gives rise to strangeness violating weak interaction processes, such as the leptonic decay of K^- into a muon and antineutrino.

- The Feynman diagram for this process is



8.4 The GIM Mechanism

- Although there are charged weak interactions that violate strangeness conservation, there are no known neutral weak interactions which violate strangeness.

- For example, the K^0 does not decay into a muon pair or two neutrinos (branching ratio $<\ 10^{-5}$).

- This means that the the $Z-$boson only interacts with quarks of the same flavour.
- We can see this by noting that the $Z-$boson interaction terms are unaffected by a unitary transformation.

- The $Z-$boson interactions with $d-$ and $s-$ quarks are proportional to

$$\overline{\tilde{d}}\,\tilde{d}+\overline{\tilde{s}}\,\tilde{s}.$$

(we have suppressed the $\gamma-$matrices which act between the fermion fields). Writing this out in terms of the physical quarks we get

$$\cos^2\theta_C\,\overline{d}\,d\;+\;\sin\theta_C\,\cos\theta_C\,\overline{s}\,d\;+\;\cos\theta_C\,\sin\theta_C\,\overline{d}\,s\;+\;\sin^2\theta_C\,\overline{s}\,s$$

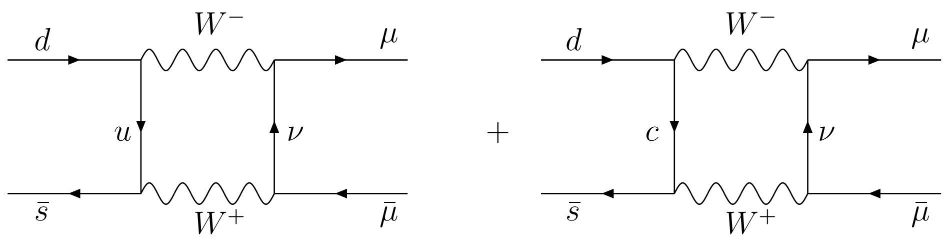
$$+\;\cos^2\theta_C\,\overline{s}\,s\;-\;\sin\theta_C\,\cos\theta_C\,\overline{d}\,s\;-\;\cos\theta_C\,\sin\theta_C\,\overline{s}\,d\;+\;\sin^2\theta_C\,\overline{d}\,d.$$

- We see that the cross-terms cancel out and we are left with simply

$$\overline{d}\,d\;+\;\overline{s}\,s.$$

- This cancellation is known as the "GIM" (Glashow-Iliopoulous-Maiani) mechanism. It was used to predict the existence of the $c-$quark.

- There can be a small contribution to strangeness changing neutral processes from higher order corrections in which we do not exchange a $Z-$boson, but two charged $W-$bosons.
- The Feynman diagrams for such a contribution to the leptonic decay of a K^0 (which consists of a $d-$quark and an $s-$antiquark) are



- These diagrams differ in the flavour of the internal quark which is exchanged, being a $u-$quark in the first diagram and a $c-$quark in the second.
- Both of these diagrams are allowed because of the Cabibbo mixing.

- The first of these diagrams gives a contribution proportional to

$$+\sin\theta_C\,\cos\theta_C,$$

which arises from the product of the two couplings involving the emission of the $W-$bosons.

- The second diagram gives a term proportional to

$$-\cos\theta_C\,\sin\theta_C.$$

- If the $c-$quark and $u-$quark had identical mass then these two contributions would cancel precisely.

- However, the $c-$quark is much more massive than the $u-$quark, there is some residual contribution.

- This was used to limit the mass of the $c-$quark to < 5 GeV, before it was discovered.

8.5 Adding a Third Generation

- Adding a third generation is achieved in the same way.

- In this case the three weak isodoublets of left-handed fermions are

$$\begin{pmatrix} u \\ \tilde{d} \end{pmatrix}, \quad \begin{pmatrix} c \\ \tilde{s} \end{pmatrix}, \quad \begin{pmatrix} t \\ \tilde{b} \end{pmatrix},$$

where $\tilde{d}$, $\tilde{s}$ and $\tilde{b}$ are related to the physical $d-$, $s-$ and $b-$quarks by

$$\begin{pmatrix} \tilde{d} \\ \tilde{s} \\ \tilde{b} \end{pmatrix} = \mathbf{V_{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{8.8}$$

- The 3×3 unitary matrix $\mathbf{V_{CKM}}$ is the "Cabibbo-Kobayashi-Maskawa (CKM) matrix".

- Once again it only affects the charged weak processes in which a $W-$boson is exchanged.

- For this reason the elements are written as

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

- A 3×3 unitary matrix can have nine independent parameters (counting the real and imaginary parts of a complex element as two parameters).

- In this case there are six possible fermions involved in the charged weak processes and so we can have five relative phase transformations, thereby absorbing five of the nine parameters.

- This means that whereas the Cabibbo matrix only has one parameter (the Cabibbo angle, θ_C) the CKM matrix has four independent parameters.

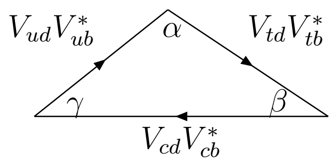
- The four independent parameters can be thought of as three mixing angles between the three pairs of generations and a complex phase.

- The requirement of unitarity puts various constraints on the elements of the CKM matrix.

- For example we have

$$V_{ud}\, V_{ub}^* \, + \, V_{cd}\, V_{cb}^* \, + \, V_{td}\, V_{tb}^* \; = \; 0$$

- This can be represented as a triangle in the complex plane known as the "unitarity triangle"



- The angles of the triangle are related to ratios of elements of the CKM matrix

$$\alpha \;=\; -\arg\left\{\frac{V_{td}\, V_{tb}^*}{V_{ud}\, V_{ub}^*}\right\} \tag{8.9}$$

$$\beta \;=\; -\arg\left\{\frac{V_{td}\, V_{tb}^*}{V_{cd}\, V_{cb}^*}\right\} \tag{8.10}$$

$$\gamma \;=\; -\arg\left\{\frac{V_{ud}\, V_{ub}^*}{V_{cd}\, V_{cb}^*}\right\} \tag{8.11}$$

- A popular representation of the CKM matrix is due to Wolfenstein and uses parameters A, which is assumed to be of order unity, a complex number, $(\rho \, + \, i\,\eta)$ and a small number λ, which is approximately equal to $\sin\theta_C$.

- In terms of these parameters the CKM matrix is written

$$\mathbf{V_{CKM}} \;=\; \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\,\lambda^3\,(\rho - i\,\eta) \\ -\lambda & 1-\lambda^2/2 & A\,\lambda^2 \\ A\lambda^3\,(1-\rho-i\,\eta) & -A\,\lambda^2 & 1 \end{pmatrix} \;+\; \mathcal{O}(\lambda^4). \tag{8.12}$$

- We see that whereas the $W-$bosons can mediate a transition between a $u-$quark and a $b-$quark (V_{ub}) or between a $t-$quark and a $d-$quark (V_{td}), the amplitude for such transitions are suppressed as the cube of the small quantity which determines the amplitude for transitions between the first and second generations, λ.

- The $\mathcal{O}(\lambda^4)$ corrections are needed to ensure the unitarity of the CKM matrix and these corrections have several matrix elements which are complex.

8.6 Mass generation - a summary

- Starting from the Yukawa interactions

$$\mathcal{L}_Y^{(q,e)} = \lambda_{uij}\overline{q}_{Li}u_{Rj}H_2 + \lambda_{dij}\overline{q}_{Li}d_{Rj}H_1 + \lambda_{eij}\overline{l}_{Li}e_{Rj}H_1 \;+\; h.c.\;, \tag{8.13}$$

where $H_2 = i\sigma_2 H_1^\dagger$. Setting the vacuum expectation value of the Higgs fields as, $\langle H_1\rangle = (0, v/\sqrt{2})^T$

$$\mathcal{L}_M^{(q,e)} = m_{uij}\overline{u}_{Li}u_{Rj} + m_{dij}\overline{d}_{Li}d_{Rj} + m_{eij}\overline{e}_{Li}e_{Rj} \;+\; h.c.\;, \tag{8.14}$$

with the mass matrices for up-quarks, down-quarks and charged leptons,

$$m_u = \lambda_u\frac{v}{\sqrt{2}}\;,\quad m_d = \lambda_d\frac{v}{\sqrt{2}}\;,\quad m_e = \lambda_e\frac{v}{\sqrt{2}}\;. \tag{8.15}$$

- The mass matrices are diagonalized by bi-unitary transformations,

$$V^{(u)\dagger}m_u\widetilde{V}^{(u)} = m_u^{diag}\;,\quad V^{(d)\dagger}m_d\widetilde{V}^{(d)} = m_d^{diag}\;,\quad V^{(e)\dagger}m_e\widetilde{V}^{(e)} = m_e^{diag}\;, \tag{8.16}$$

with $V^{(u)\dagger}V^{(u)} = 1$, etc. , which also define the transition from *mass* eigenstates $\psi_{L\alpha}$, $\psi_{R\alpha}$ to *weak* eigenstates ψ_{Li}, ψ_{Ri},

$$u_{Li} = V_{i\alpha}^{(u)}u_{L\alpha}\;,\quad d_{Li} = V_{i\alpha}^{(d)}d_{L\alpha}\;,\ldots, e_{Ri} = \widetilde{V}_{i\alpha}^{(e)}e_{R\alpha}\;. \tag{8.17}$$

- Since in general the transformation matrices are different for up- and down-quarks one obtains a mixing between mass eigenstates in the charged current (CC) weak interactions,

$$\begin{aligned}\mathcal{L}_{EW}^{(q)} &= -\frac{g}{\sqrt{2}}\sum_i \overline{u}_{Li}\gamma^\mu d_{Li}\;W_\mu^+ \;+\ldots \\ &= -\frac{g}{\sqrt{2}}\sum_{\alpha,\beta} \overline{u}_{L\alpha}\gamma^\mu V_{\alpha\beta}d_{L\beta}\;W_\mu^+ \;+\ldots\;,\end{aligned} \tag{8.18}$$

where $V_{\alpha\beta} = V_{\alpha i}^{(u)\dagger}V_{i\beta}^{(d)}$ is the familiar CKM mixing matrix.

- In general the Yukawa couplings are complex; hence, also the CKM matrix is complex, which leads to CP violation in weak interactions.

- Without right-handed neutrinos, weak and mass eigenstates can always be chosen to coincide for leptons.

- As a consequence, there is no mixing in the leptonic charged current,

$$\mathcal{L}_{EW}^{(l)} = -\frac{g}{\sqrt{2}} \sum_{\alpha} \overline{e}_{Li} \gamma^{\mu} \nu_{Li} \; W_{\mu}^{-} \; + \ldots \; , \tag{8.19}$$

  and electron-, muon- and tau-number are separately conserved.

- On the contrary, in the quark sector only the total baryon number is conserved.

- We can write the full charged current interactions in terms of currents are

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}} \left(J_{\mu}^{-} W^{+\mu} + J_{\mu}^{+} W^{-\mu} \right)$$

  where

$$J_{\mu}^{+} = \frac{1}{2} \overline{u}_{\alpha} V_{\alpha\beta} \gamma_{\mu} (1 - \gamma_5) d_{\beta} + \frac{1}{2} \overline{\nu}_{\alpha} \gamma_{\mu} (1 - \gamma_5) e_{\alpha}$$

  and

$$J_{\mu}^{-} = \frac{1}{2} \overline{d}_{\alpha} V_{\alpha\beta}^{\dagger} \gamma_{\mu} (1 - \gamma_5) u_{\beta} + \frac{1}{2} \overline{e}_{\alpha} \gamma_{\mu} (1 - \gamma_5) \nu_{\alpha}$$

- The neutral current interaction does not contain the CKM matrix and can be written in the form

$$\begin{aligned} \mathcal{L}_{nc} &= \sum_{f_i} -\frac{g}{2\cos\theta_W} \overline{f_i} \left(t^3(f_i)\, \gamma^{\mu} \,(1 - \gamma_5) - \; 2Q(f_i) \sin^2 \theta_W \gamma^{\mu} \right) \; f_i \, Z_{\mu} - Q(f_i) \, g \, \sin\theta_W \, \overline{f_i} \, \gamma^{\mu} \, f_i \, A_{\mu} \\ &= -\frac{g}{2\cos\theta_W} J_{\mu}^{0} \, Z^{\mu} - \; g \, \sin\theta_W \, J_{\mu}^{\mathrm{em}} A^{\mu} \end{aligned}$$

  where

$$J_{\mu}^{0} = J_{\mu}^{3} - \sin^2 \theta_W \, J_{\mu}^{\mathrm{em}}$$

  and

$$J_{\mu}^{3} = \sum_{f_i} \overline{f_i} \; \left(t^3(f_i) \, \gamma_{\mu} (1 - \gamma_5) \right) f_i$$

  and

$$J_{\mu}^{\mathrm{em}} = \sum_{f_i} Q(f_i) \, g \, \sin\theta_W \, \overline{f_i} \, \gamma_{\mu} \, f_i$$

- Often the neutral current is written

$$
\begin{aligned}
J^0_\mu &= \sum_{f_i} \left[g_L^{f_i} \overline{f}_{iL} \gamma_\mu f_{iL} + g_R^{f_i} \overline{f}_{iR} \gamma_\mu f_{iR} \right] \\
&= \frac{1}{2} \sum_{f_i} \left[g_L^{f_i} \overline{f}_i \gamma_\mu (1 - \gamma_5) f_i + g_R^{f_i} \overline{f}_i \gamma_\mu (1 + \gamma_5) f_i \right]
\end{aligned}
$$

where

$$g^{f_i}_{L,R} = t_3(f_i) - Q(f_i) \sin^2 \theta_W$$

- At low energies the effective action has the form

$$
\begin{aligned}
\mathcal{L}_{\text{eff}} &= -\frac{g^2}{2M_W^2} J^+_\mu J^{-\mu} - \frac{g^2}{2\cos^2\theta_W M_Z^2} J^0_\mu J^{0\mu} \\
&= -\frac{g^2}{2M_W^2} J^+_\mu J^{-\mu} - \frac{g^2}{2M_W^2} J^0_\mu J^{0\mu}
\end{aligned}
\tag{8.20}
$$

which can be compared with Fermi theory to get the value

$$G_F = \frac{g^2}{4\sqrt{2}\, M_W^2}$$

as we did before.

8.7 CP Violation

- The possibility that some of the elements of the CKM matrix may be complex provides a mechanism for the violation of CP conservation.
- Violation of CP conservation has been observed in the K^0, $\overline{K}^0$ system, and is currently being investigated for $B-$mesons.

- Higher order corrections to the masses of B^0 and $\overline{B^0}$, give rise to mixing between the two states.
- Thus the mass matrix can be written

$$\begin{pmatrix} M_{B^0} & \Delta M \\ (\Delta M)^* & M_{B^0} \end{pmatrix}.$$

- The mass eigenstates are therefore

$$|B_L\rangle = p\,|B^0\rangle + q\,|\overline{B}^0\rangle, \tag{8.21}$$

whose mass is $M_{B^0} - \frac{1}{2}\Delta m$ and

$$|B_H\rangle \;=\; p\,|B^0\rangle \;-\; q\,|\overline{B}^0\rangle, \tag{8.22}$$

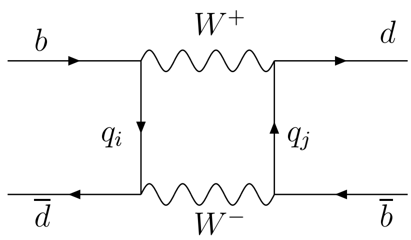
whose mass is $M_{B^0} + \frac{1}{2}\Delta m$.

- If ΔM were real then we would have $p = q = 1/\sqrt{2}$ and these mass eigenstates would be CP eigenstates, using the fact that

$$CP|B^0\rangle \;=\; -|\overline{B}^0\rangle.$$

- However, the non-zero phases in the CKM matrix give rise to a complex phase for ΔM, so that the ratio of p and q is a complex phase, indicating that B_L and B_H are *not* CP eigenstates.

A typical weak interaction contribution to the mass-mixing term, ΔM is given by the Feynman diagram



- Note that on the left we have a B^0, consisting of a $b-$quark and an $d-$antiquark, whereas on the right we have a $\overline{B}^0$ consisting of an $d-$quark and a $b-$antiquark.

- The internal quarks marked q_i and q_j can each be $u-$, $c-$ or $t-$quarks and each of the vertices carries some element of the CKM matrix.

- The total contribution, therefore, may be written

$$\sum_{i=u,c,t}\;\sum_{j=u,c,t} V_{ib}\,V_{id}^*\,V_{jb}\,V_{jd}^*\,a_{ij}.$$

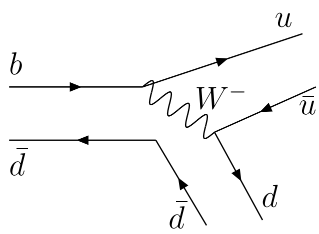
- Once again, if all the masses of the quarks were equal then the amplitudes a_{ij} would all be equal, and this would vanish by the unitarity constraints imposed on the elements V_{ik}.

- Since the quarks do not all have the same mass, there is some residual contribution.

- Diagram is dominated by the term in which a $t-$quark is exchanged on both sides, since this quark is much more massive than the rest.

- Restricting ourselves to the $t-$quark exchange contribution, we can read off the phase of this contribution, without calculating the diagram itself.
- It is given by the phase of the products, of the CKM matrix elements entering in the diagram, namely
$$(V_{td}^* V_{tb})^2.$$
- The phase of this quantity is the ratio of p and q, so we have
$$\frac{p}{q} \;=\; \frac{V_{td}^* V_{tb}}{V_{td} V_{tb}^*}$$
- Now suppose that at time $t = 0$ we prepare a state which is purely B^0.
- Accounting for the fact that the B^0 meson has a decay rate Γ, we can use eqs.(8.21, 8.22) to write the state at time t as
$$|B(t)\rangle \;=\; e^{-iMt}\, e^{-\Gamma t/2} \left(\cos\left(\frac{\Delta m}{2} t\right) |B^0\rangle \;+\; i\frac{q}{p} \sin\left(\frac{\Delta m}{2} t\right) |\overline{B}^0\rangle . \right) \tag{8.23}$$
- Now suppose that the amplitude for a state $|B^0\rangle$ to decay into some CP eigenstate $|f\rangle$ is A_f, whereas the amplitude for a state $|\overline{B}^0\rangle$ to decay into the state $|f\rangle$ is $\overline{A}_f$.
- Once again, if CP were conserved, we would have
$$A_f \;=\; \pm \overline{A}_f,$$
but the CP violating phases give rise to a more general complex phase for the ratio of these two amplitudes.
- This means that the amplitude to find the state $|f\rangle$ after time t is given by
$$\langle f|\mathcal{H}|B(t)\rangle \;=\; e^{-iMt}\, e^{-\Gamma t/2} \left(\cos\left(\frac{\Delta m}{2} t\right) A_f \;+\; i\frac{q}{p} \sin\left(\frac{\Delta m}{2} t\right) \overline{A}_f \right). \tag{8.24}$$
- Similarly, if we had prepared a $\overline{B}^0$ at $t = 0$ the amplitude to find the state $|f\rangle$ would be
$$\langle f|\mathcal{H}|\overline{B}(t)\rangle \;=\; e^{-iMt}\, e^{-\Gamma t/2} \left(\cos\left(\frac{\Delta m}{2} t\right) \overline{A}_f \;+\; i\frac{p}{q} \sin\left(\frac{\Delta m}{2} t\right) A_f \right). \tag{8.25}$$
- Taking the square moduli, to find the decay rates we arrive at the result
$$\frac{\Gamma(B(t) \to f) - \Gamma(\overline{B}(t) \to f)}{\Gamma(B(t) \to f) + \Gamma(\overline{B}(t) \to f)} \;=\; -\; \sin(\Delta m\, t)\, \Im m \left(\frac{q}{p} \frac{\overline{A}_f}{A_f} \right). \tag{8.26}$$

- For example, if the state $|f\rangle$ is the CP even two-pion state $|\pi^0\,\pi^0\rangle$, the Feynman diagram at the quark level for $A_{2\pi}$ is



- To calculate the decay amplitudes we would need to know the wavefunctions for the mesons in terms of the constituent quark-antiquark pairs, but for the ratio $\overline{A}_{2\pi}/A_{2\pi}$ we just need the ratios of the CKM matrix elements occurring in this diagram, namely

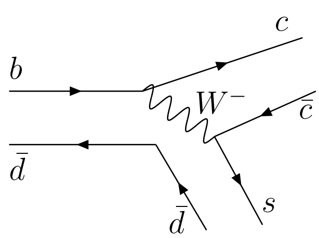
$$\frac{\overline{A}_{2\pi}}{A_{2\pi}} = \frac{V_{ub}\,V_{ud}^*}{V_{ub}^*\,V_{ud}},$$

so that (using eq.(8.9))

$$\Im m\left(\frac{q}{p}\frac{\overline{A}_{2\pi}}{A_{2\pi}}\right) = \Im m\left(\frac{V_{td}\,V_{tb}^*}{V_{td}^*\,V_{tb}}\frac{V_{ub}\,V_{ud}^*}{V_{ub}^*\,V_{ud}}\right) = -\sin{(2\alpha)} \qquad (8.27)$$

- As a further example, we consider the so-called "golden channel" where $|f\rangle$ is the state $|J/\psi\,K_S\rangle$.

- In this case the quark level Feynman diagram is



- Here there is a further complication since the outgoing state $(s\,\bar{d})$ is actually $\overline{K}^0$ (and likewise for the $\overline{B}^0$ decay it would be a K^0.

- As in the B^0 system, the mass eigenstates are given by

$$|K_S\rangle \;=\; p_K|K^0\rangle + q_K|\overline{K}^0\rangle$$
$$|K_L\rangle \;=\; p_K|K^0\rangle - q_K|\overline{K}^0\rangle.$$

- Once again, if CP were conserved we would have $p_K = q_K = 1/\sqrt{2}$, and these mass eigenstates would be eigenstates of CP.

- The phases in the CKM matrix introduce a phase in the ratio of p_K and q_K, calculated from diagrams similar to the ones for the B^0 system (but with the $b-$quark replaced by an $s-$quark).

- In this case it is the diagram with internal c-quark exchange that dominates (although the mass of the$c-$ quark is much less than the $t-$quark the CKM matrix elements are much larger for c-quark exchange than $t-$quark exchange and this effect dominates), so we have a factor

$$\frac{q_K}{p_K} \;=\; \frac{V_{cd}^*\,V_{cs}}{V_{cd}\,V_{cs}^*}$$

which enters in the ratio of the decay amplitudes, giving

$$\frac{\overline{A}_{J/\psi\,K_S}}{A_{J/\psi\,K_S}} \;=\; -\frac{V_{cb}\,V_{cs}^*}{V_{cb}^*\,V_{cs}}\frac{V_{cd}^*\,V_{cs}}{V_{cd}\,V_{cs}^*} \;=\; -\frac{V_{cb}\,V_{cd}^*}{V_{cb}^*\,V_{cd}},$$

(a minus occurs because the $J/\psi\,K_S$ state is CP odd) so that (using eq.(8.10))

$$\Im m\left(\frac{q}{p}\frac{\overline{A}_{J/\psi\,K_S}}{A_{J/\psi\,K_S}}\right) \;=\; -\Im m\left(\frac{V_{td}\,V_{tb}^*}{V_{td}^*\,V_{tb}}\frac{V_{cb}\,V_{cd}^*}{V_{cb}^*\,V_{cd}}\right) \;=\; \sin(2\beta)\,. \tag{8.28}$$

8.8 Summary

- A left-handed quark doublet has weak hypercharge $Y = \frac{1}{6}$, and the right-handed components have weak hypercharges equal to the electric charges of the quarks themselves.

- In order to generate a mass for the $T^3 = +\frac{1}{2}$ quark, one has to add a different type of Yukawa coupling given by (8.4).

- A second generation may be added as a copy, but in this case one can have mass-mixing between quarks of different generations. In terms of the mass eigenstates, the charged $W-$bosons mediate transitions between a $T^3 = +\frac{1}{2}$ quark (u or c) and a superposition of $T^3 = -\frac{1}{2}$ quarks (d and s). This mechanism allows weak interactions which violate strangeness conservation. The mixing matrix has only one independent parameter, the Cabibbo angle.

- The GIM mechanism guarantees that there are no strangeness changing neutral processes. Weak interactions involving the exchange of a $Z-$boson do not change flavour. There is a small violation of this in higher orders owing to the mass splitting between the quarks.

- When one includes a third generation, the mixing matrix for the $T^3 = -\frac{1}{2}$ quarks (d, s and b) is the CKM matrix. This matrix has has four independent parameters, so that some of the matrix elements may be complex.

- The possibility that some of the elements of the CKM matrix may be complex leads to a weak interaction contribution to a mass mixing of B^0 and $\overline{B}^0$ which can be complex. This gives rise to CP violation, since the eigenstates of the B^0 mass matrix are then no longer also eigenstates of CP. The CKM matrix also introduces phases in the ratios of the decay amplitudes for B^0 and $\overline{B}^0$ to a given CP eigenstate. Products of the phase of the mass mixing and the ratio of the decay amplitudes can be observed directly in tagged $B-$meson experiments and the angles α and β of the unitarity triangle can be directly measured.

9 Neutrinos and the MNS matrix

9.1 Dirac Neutrino masses and mixings

- The simplest way to introduce

$$\mathcal{L}_Y^{(\nu)} = \lambda_{\nu ij} \bar{l}_{Li} \nu_{Rj} H_2 + \ h.c. \tag{9.1}$$

- The Yukawa interaction defines the quantum numbers of the right-handed neutrino: it carries lepton number, which is a global charge, but no colour, weak isospin or hypercharge

$$\nu_R : \qquad (1, 1, 0)$$

- The Yukawa interaction which couples left-handed and right-handed neutrinos yields after spontaneous symmetry breaking the Dirac neutrino mass matrix

$$m_D = \frac{\lambda_\nu v}{\sqrt{2}}$$

- with $v = 246$ GeV we need

$$\lambda_\nu \sim 10^{-13}$$

  such that

$$m_D \sim 0.01 \text{ eV}$$

  or smaller.

- Problem with this is that we must justify the size of λ_ν.

9.2 Majorana Neutrinos and the see-saw mechanism

- A popular way to introduce neutrino masses is to allow the ν_Rs to be majorana.

- This allows additional Yukawa couplings and a Majorana mass term,

$$\mathcal{L}_Y^{(\nu)} = \lambda_{\nu ij} \bar{l}_{Li} \nu_{Rj} H_2 + \frac{1}{2} M_{ij} \overline{\nu^c_{Ri}} \nu_{Rj} \ + \ h.c. \tag{9.2}$$

- A Majorana mass term is allowed for the right-handed neutrinos, consistent with the gauge symmetries of the theory.

- It is very important that these masses are *not* generated by the Higgs mechanism and can therefore be much larger than ordinary quark and lepton masses.

- This leads to light neutrino masses via the seesaw mechanism.

- The Yukawa interaction which couples left-handed and right-handed neutrinos yields after spontaneous symmetry breaking the Dirac neutrino mass matrix $m_D = h_\nu v/\sqrt{2}$, the complete mass terms are given by

$$\mathcal{L}_M^{(\nu)} = m_{Dij}\overline{\nu_{L}}_i\nu_{Rj} + \frac{1}{2}M_{ij}\overline{\nu^c_{R}}_i\nu_{Rj} \ + \ h.c. \tag{9.3}$$

- We now wish to write the term as a mixing matrix (and its hermitian conjugate) to do so we need the following

$$\overline{\nu_L}m_D\nu_R = \frac{1}{2}\left(\overline{\nu_L}m_D\nu_R - \nu_R^T m_D^T \overline{\nu_L}^T\right)$$

then using the following

$$\begin{array}{lccc} & \nu_L^c = C(\overline{\nu_L})^T & \rightarrow & C^{-1}\nu_L^c = (\overline{\nu_L})^T \\ \text{and} & \overline{\nu_L^c} = -\nu_L^T C^{-1} & \rightarrow & \overline{\nu_L^c}C = -\nu_L^T \end{array}$$

with

$$C = i\gamma_2\gamma_0 \qquad C^\dagger = C^{-1} \qquad C^T = -C$$

using these identities we then have

$$-\nu_R^T m_D^T \overline{\nu_L}^T = \overline{\nu_R^c} C m_D^T C^{-1}\nu_L^c = \overline{\nu_R^c} m_D^T \nu_L^c$$

the mass terms are then

$$\mathcal{L}_M^{(\nu)} = \frac{1}{2}m_{Dij}\overline{\nu_{L}}_i\nu_{Rj} + \frac{1}{2}m_{Dij}^T\overline{\nu^c_{R}}_i\nu^c_{Lj} + \frac{1}{2}M_{ij}\overline{\nu^c_{R}}_i\nu_{Rj} \ + \ h.c.$$

and can be written in matrix form

$$\mathcal{L}_M^{(\nu)} = \frac{1}{2}\begin{pmatrix} \overline{\nu_L} & \overline{\nu_R^c} \end{pmatrix}\begin{pmatrix} 0 & m_D \\ m_D^T & M \end{pmatrix}\begin{pmatrix} \nu_L^c \\ \nu_R \end{pmatrix} \ + \ h.c. \tag{9.4}$$

- The unitary matrix which diagonalizes this mass matrix is easily constructed as power series in $\xi = m_D/M$. Up to terms $\mathcal{O}(\xi^3)$ one obtains

$$\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\xi\xi^\dagger & \xi \\ -\xi^\dagger & 1 - \frac{1}{2}\xi\xi^\dagger \end{pmatrix}\begin{pmatrix} L \\ R^c \end{pmatrix} . \tag{9.5}$$

- In terms of the new left- and right-handed fields, L and R, the mass matrix is diagonal,

$$\mathcal{L}_M^{(\nu)} = \frac{1}{2}\begin{pmatrix} \overline{L} & \overline{R^c} \end{pmatrix}\begin{pmatrix} m_\nu & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} L^c \\ R \end{pmatrix} \ + \ h.c. \tag{9.6}$$

- Here the mass matrix m_ν is given by

$$m_\nu = -m_D \frac{1}{M} m_D^T \; . \tag{9.7}$$

- This is the famous seesaw mass relation.
- With $M \gg m_D$, one obviously has $m_\nu \ll m_D$.
- As an example, consider the case of just one generation.
- Choosing for m_D the largest know fermion mass,

$$m_D \sim m_t \sim 100 \; GeV$$

  and for M the unification scale of unified theories,

$$M \sim 10^{15} \; GeV$$

  one finds

$$m_\nu \sim 10^{-2} \; eV$$

  which is precisely in the range of present experimental indications for neutrino masses.

- m_ν and M are the mass matrices of the light neutrinos and their heavy partners, respectively.
- The corresponding mass eigenstates are Majorana fermions,

$$\nu = L + L^c = \nu^c \; , \quad N = R^c + R = N^c \; . \tag{9.8}$$

- L, R^c and L^c, R are the corresponding left- and right-handed components,

$$L = \frac{1-\gamma_5}{2}\nu \; , \quad R^c = \frac{1-\gamma_5}{2} N \; , \quad \text{etc.} \tag{9.9}$$

- In terms of the Majorana fields the mass terms read,

$$\begin{aligned} \mathcal{L}_M^{(\nu)} &= \frac{1}{2}\overline{\nu} m_\nu \frac{1+\gamma_5}{2}\nu + \frac{1}{2}\overline{N} m_\nu \frac{1+\gamma_5}{2} N \; + \; h.c. \\ &= \frac{1}{2}\overline{\nu}\left(\mathrm{Re}\{m_\nu\} + i\mathrm{Im}\{m_\nu\}\gamma_5\right)\nu + \frac{1}{2}\overline{N}\left(\mathrm{Re}\{M\} + i\mathrm{Im}\{M\}\gamma_5\right) N \; . \end{aligned} \tag{9.10}$$

- Note, that in general the mass matrices have real and imaginary parts. This is a consequence of complex Yukawa couplings and a possible source of CP violation.

9.3 MNS matrix and Counting phases for Dirac and Majorana Neutrinos

- First let us recall in detail what happens for the **quarks**
- In the mass basis we have the following terms in the lagrangian

$$\mathcal{L}_{\text{quarks}} = -\frac{g}{\sqrt{2}} \sum_{\alpha,\beta} \overline{u}_{L\alpha} \gamma^{\mu} V_{\alpha\beta} d_{L\beta} \; W_{\mu}^{+} \; + m_u^{\text{diag}} (\overline{u_L} u_R + \overline{u_R} u_L) + m_d^{\text{diag}} (\overline{d_L} d_R + \overline{d_R} d_L) + \dots \; ,$$

- The CKM matrix is a 3X3 Unitary matrix.
- A general unitary matrix has 9 real parameters
- We can remove phases be performing field redefintions
- We have 6 fields and so we can remove 5 parameters leaving 4 real parameters: 3 angles and 1 phase

Now Dirac Neutrinos

- Neutrino masses imply mixing in the leptonic charged current.
- Counting is the same as for quarks
- Lagrangian mass terms are:

$$\mathcal{L}_{\text{leptonmass}} = m_{Dij} \overline{\nu_L} \nu_R + m_{eij} \overline{e_{Li}} e_{Rj} + \text{h.c.} \; , \tag{9.11}$$

- The mass mixing matrices are diagonalised by the following

$$\begin{aligned} V^{(\nu)\dagger} m_D \widetilde{V}^{(\nu)} &= m_{\nu}^{diag} \; , \\ V^{(e)\dagger} m_e \widetilde{V}^{(e)} &= m_e^{diag} \; , \end{aligned} \tag{9.12}$$

- with the mass and weak eigenstates related as

$$\nu_{Li} = V_{i\alpha}^{(\nu)} u_{L\alpha} \; , \quad \nu_{Ri} = \widetilde{V}_{i\alpha}^{(\nu)} \nu_{R\alpha} \; , \quad e_{Li} = V_{i\alpha}^{(e)} e_{L\alpha} \; , \quad e_{Ri} = \widetilde{V}_{i\alpha}^{(e)} e_{R\alpha} \; .$$

- The resulting weak interaction terms in the mass eigenstate basis

$$\mathcal{L}_{\text{leptonmass}} = -\frac{g}{\sqrt{2}} \sum_{\alpha,\beta} \overline{\nu}_{L\alpha} \gamma^{\mu} V_{\alpha\beta}^{mns} e_{L\beta} \; W_{\mu}^{+} \; + m_{\nu}^{\text{diag}} (\overline{\nu_L} \nu_R + \overline{\nu_R} \nu_L) + m_e^{\text{diag}} (\overline{e_L} e_R + \overline{e_R} e_L) + \dots \; ,$$

where

$$V_{\alpha\beta}^{mns} = V_{\alpha i}^{(\nu)\dagger} V_{i\beta}^{(e)}$$

- We can do the same counting as with the CKM matrix:
- We have can perform 6 field redefinitions removing 5 parameter leaving 4: 3 angles and a phase
- Note: due to the mixing e-, μ- and τ-number are now no longer separately conserved.

Now Majorana Neutrinos

- The Majorana mass terms from before are

$$\mathcal{L}_M^{(\nu)} = \overline{\nu} m_\nu \nu + \overline{N} M N \; . \tag{9.13}$$

- Forget the heavy neutrinos
- If the neutrino mass matrix is diagonalized by the unitary matrix $U^{(\nu)}$,

$$U^{(\nu)\dagger} m_\nu U^{(\nu)*} = - \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \; , \tag{9.14}$$

the mixing matrix U appears in the charged current,

$$\mathcal{L}_{EW}^{(l)} = - - \frac{g}{\sqrt{2}} \sum_{\alpha,\beta} \overline{\nu}_{L\alpha} \gamma^\mu U_{\alpha\beta}^{mns} e_{L\beta} \; W_\mu^+ \; + \overline{\nu} m_\nu \nu + \overline{N} M N + \dots \; , \tag{9.15}$$

where

$$U_{ij}^{mns} = U_{i\alpha}^{(\nu)\dagger} V_{\alpha j}^{(e)} \; . \tag{9.16}$$

again e-, μ- and τ-number are now no longer separately conserved.

- This time the counting is different.
- We do not have the same freedom as we did before to make field redefinitions due to the Majorana nature of the neutrinos.
- We can rewrite the Majorana mass terms as follow

$$\overline{\nu} m_\nu \nu = -\nu^T C^{-1} m_\nu \nu$$

- If for example we make a phase redefinition of one of the neutrinos fields, the mass term will change and pick up a phase
- This means that we can remove fewer parameters from the MNS matrix meaning we have more physical parameters.

- Turns out the result is that you can remove 2 less phases
- In Majorana neutrino case the MNS matrix we have 3 angles and 3 phases.
- 2 extra phases are often referred to as Majorana phases

10 Anomalies

- So far, we have taken the hypercharge assignments of our fermion fields but at the classical level the assignments are arbitrary.
- However, at the quantum level this is not true.
- The Adler-Bell-Jackiw anomaly spoils the renormalisabilty of the theory whenever it spoils the conservation of a gauged current
- Adler-Bardeen theorem asserts that the anomaly occurs only at one loop order.
  $\Rightarrow$ Only need to make sure the relevant one loop diagrams vanish to leave the theory renormalisable
- This restricts the hypercharge assignments (and also the particle content of the theory)
- We also find that there is an anomaly associated with the chiral symmetry and we start by analysing this.

Axial Current and triangle diagrams

- For massless fermions, the 4d Dirac equation splits into separate terms for left and right handed fermions
- Introducing gauge field we replace derivatives by covariant derivatives
  $\rightarrow$ does not affect this separation
- This should mean that the vector and axial vector currents remain conserved
  BUT this is not the case for the axial current after we include quantum effects

- Without the quantum effects we have

$$j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x) \qquad j^{\mu 5}(x) = \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)$$

$$\partial_\mu j^\mu = 0, \qquad \partial_\mu j^{\mu 5} = 0$$

  in the limit of zero quark masses and we are considering the following symmetry transformations

$$\psi(x) \to e^{i\alpha}\psi \qquad \psi(x) \to e^{i\alpha\gamma_5}\psi$$

  for vector and axial symmetries
  What happens in the quantum case, start with QED Lagrangian with one massless fermion

$$\mathcal{L} = \overline{\psi}(i\not{D})\psi - \frac{1}{4}(F_{\mu\nu})^2$$

- To find out what happens to the axial vector current we can study the operator equation for the divergence of $j^{\mu L}$.

- First solve for the equations of motion for the fermion field

$$\partial\!\!\!/\psi = -ie A\!\!\!/\psi, \qquad \partial_\mu \overline{\psi}\gamma^\mu = ie\overline{\psi}A\!\!\!/$$

- The axial vector current is a composite operator built out of fermion fields.

- Products of local operators are often singular

- Solution is to place two fermion fields at distinct points separated by a distance ϵ, then later take the limit $\epsilon \to 0$.

$$j^{\mu 5} = \underset{\epsilon \to 0}{\text{symmetric limit}} \left(\overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma^5 \underbrace{\exp\left[-ie\int_{x-\epsilon/2}^{x+\epsilon/2} dz.A(z)\right]}_{\text{Wilson Line}} \psi(x-\frac{\epsilon}{2})\right)$$

- We have to introduce the Wilson line to maintain gauge invariance.

- To ensure $j^{\mu 5}$ has the correct Lorentz transformation properties, the limit $\epsilon \to 0$ should be taken symmetrically

$$\underset{\epsilon \to 0}{\text{symm limit}} \left(\frac{\epsilon^\mu}{\epsilon^2}\right) = 0, \qquad \underset{\epsilon \to 0}{\text{symm limit}} \left(\frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2}\right) = \frac{1}{4}g^{\mu\nu}$$

Now calculate the divergence of this current

- Acting with the partial derivative

$$\begin{aligned}
\partial_\mu j^{\mu 5} &= \underset{\epsilon \to 0}{\text{symm limit}} \Big((\partial_\mu\overline{\psi}(x+\frac{\epsilon}{2}))\gamma^\mu\gamma^5 \exp\left[-ie\int_{x-\epsilon/2}^{x+\epsilon/2} dz.A(z)\right]\psi(x-\frac{\epsilon}{2}) \\
&+ \overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma^5 \exp\left[-ie\int_{x-\epsilon/2}^{x+\epsilon/2} dz.A(z)\right]\left(\partial\!\!\!/\psi(x-\frac{\epsilon}{2})\right) \\
&+ \overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma^5\left[-ie\epsilon^\nu\partial_\mu A_\nu(x)\right]\psi(x-\frac{\epsilon}{2})\Big)
\end{aligned} \qquad (10.1)$$

where in the last term we have expanded in ϵ keeping terms up to order ϵ.

- Using the equations of motion

$$\not\partial\psi = -ie\not A\psi, \qquad \partial_\mu\overline{\psi}\gamma^\mu = ie\overline{\psi}\not A$$

we have

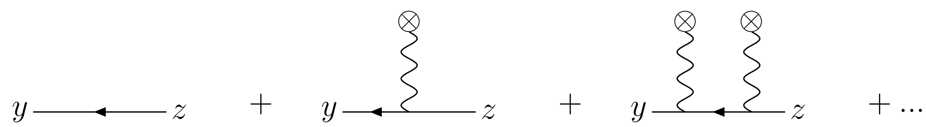
$$\begin{aligned}\partial_\mu j^{\mu 5} &= \underset{\epsilon\to 0}{\text{symm limit}} \left[\overline{\psi}(x+\frac{\epsilon}{2})\left[ie\not A(x+\frac{\epsilon}{2}) - ie\not A(x-\frac{\epsilon}{2}) - ie\epsilon^\nu\gamma^\mu\partial_\mu A_\nu(x)\right]\gamma^5\psi(x-\frac{\epsilon}{2})\right] \\ &= \underset{\epsilon\to 0}{\text{symm limit}} \left[\overline{\psi}(x+\frac{\epsilon}{2})\left[-ie\gamma^\mu\epsilon^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu)\right]\gamma^5\psi(x-\frac{\epsilon}{2})\right]\end{aligned}$$

- Looks like this goes to zero when we take $\epsilon\to 0$ but we have to take account of the singular nature of the fermion bilinear is singular

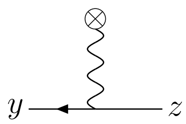
$$\begin{aligned}\psi(y)\overline{\psi}(z) &= \int\frac{d^4k}{(2\pi)^4}e^{-ik.(y-z)}\frac{i\not k}{k^2} \\ &= -\not\partial\left(\frac{i}{4\pi^2}\frac{1}{(y-z)^2}\right) \\ &= \frac{-i}{2\pi^2}\left(\frac{\gamma^\alpha(y-z)_\alpha}{(y-z)^4}\right).\end{aligned}$$

singular as $(y-z)\to 0$ but when the above is traced over with $\gamma^\mu\gamma^5$ we get zero
$\to$ this means we need to look to higher orders in the expansion of the products of operators.

- We redo the calculation with a non-zero background gauge field
- This expansion is pictorially represented as



- The first term we have already calculated, so what about the second term?



$$= \int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}e^{-i(k+p).y}e^{ik.z}\frac{i(\not k+\not p)}{(k+p)^2}(-ie\not A(p))\frac{i\not k}{k^2}.$$

This contribution leads to the expectation value

$$\begin{aligned}\langle\overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma^5\psi(x-\frac{\epsilon}{2})\rangle &= \int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\mathrm{e}^{ik.\epsilon}\mathrm{e}^{-ip.x}\mathrm{tr}\left[(-\gamma^\mu\gamma^5)\frac{i(\not k+\not p)}{(k+p)^2}(-ie\not A(p))\frac{i\not k}{k^2}\right] \\ &= \int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\mathrm{e}^{ik.\epsilon}\mathrm{e}^{-ip.x}\frac{4e\epsilon^{\mu\alpha\beta\gamma}(k+p)_\alpha A_\beta(p)k_\gamma}{k^2(k+p)^2}.\end{aligned}$$

To evaluate the limit $\epsilon\to 0$, expand the integrand for large k,

$$\begin{aligned}\langle\overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma^5\psi(x-\frac{\epsilon}{2})\rangle &\sim 4e\epsilon^{\mu\alpha\beta\gamma}\int\frac{d^4p}{(2\pi)^4}\mathrm{e}^{-ip.x}p_\alpha A_\beta(p)\int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{ik.\epsilon}\frac{k_\gamma}{k^4} \\ &= 4e\epsilon^{\mu\alpha\beta\gamma}(\partial_\alpha A_\beta(x))\frac{\partial}{\partial\epsilon^\gamma}\left(\frac{i}{16\pi^2}\log\frac{1}{\epsilon^2}\right) \\ &= 2e\epsilon^{\mu\alpha\beta\gamma}F_{\alpha\beta}(x)\left(\frac{-i}{8\pi^2}\frac{\epsilon_\gamma}{\epsilon^2}\right)\end{aligned}$$

- Now we substitute this into $\partial_\mu j^{\mu5}$ and we find

$$\begin{aligned}\partial_\mu j^{\mu5} &= \underset{\epsilon\to 0}{\text{symm limit}}\left[\overline{\psi}(x+\frac{\epsilon}{2})\left[-ie\gamma^\mu\epsilon^\nu(\partial_\mu A_\nu-\partial_\nu A_\mu)\right]\gamma^5\psi(x-\frac{\epsilon}{2})\right] \\ &= \underset{\epsilon\to 0}{\text{symm limit}}\left[\frac{e}{4\pi^2}\epsilon^{\mu\alpha\beta\gamma}F_{\alpha\beta}\left(\frac{-i\epsilon_\gamma}{\epsilon^2}\right)(-ie\epsilon^\nu F_{\mu\nu})\right] \\ &= -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}.\end{aligned}$$

- This equation expresses the non-conservation of the axial vector current and is called the **Adler-Bell-Jackiw anomaly**

- Adler and Bardeen proved that this operator relation is true to all orders in QED perturbation theory
  → **Receives no further radiative corrections**

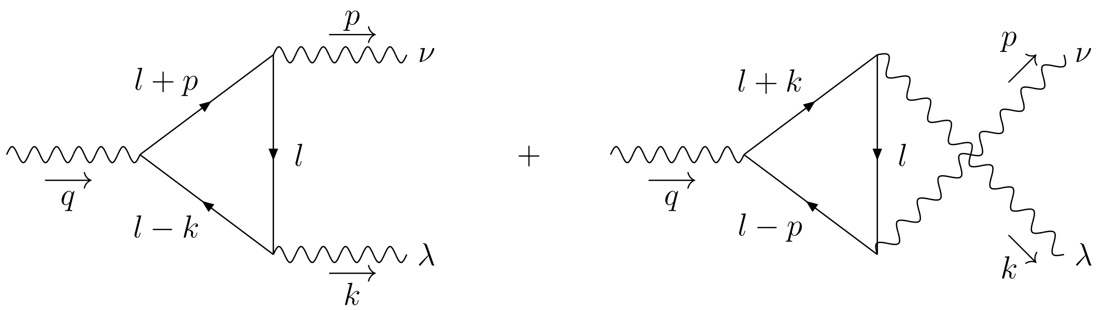
10.1 Triangle diagrams

- We can confirm the ABJ relation by checking that divergence of the axial vector current has a non-zero matrix element to create two photons

- To do so we analyze the matrix element

$$\int d^4x e^{-iq.x}\langle p,k|j^{\mu 5}(x)|0\rangle = (2\pi)^4\delta^{(4)}(p+k-q)\underbrace{\epsilon^*_\nu(p)\epsilon^*_\lambda(k)}_{\text{Polarisation vectors}}\mathcal{M}^{\mu\nu\lambda}(p,k)$$

- The leading order diagrams contributing to $M^{\mu\nu\lambda}$ are



- The first diagram gives a contribution

$$I_1 = (-1)(-ie)^2\int\frac{d^4l}{(2\pi)^4}\mathrm{tr}\left[\gamma^\mu\gamma^5\frac{i(\not{l}-\not{k})}{(l-k)^2}\gamma^\lambda\frac{i\not{l}}{l^2}\gamma^\nu\frac{i(\not{l}+\not{p})}{(l+p)^2}\right],$$

and the second diagram gives exactly the same but with (p,ν) and (k,λ) interchanged.

- Taking the divergence of the axial current in the above is equivalent to dotting this quantity with iq_μ.

- Acting on the right-hand side of I_1 and using the identity

$$q_\mu\gamma^\mu\gamma^5 = (\not{l}+\not{p}-\not{l}+\not{k})\gamma^5 = (\not{l}+\not{p})\gamma^5+\gamma^5(\not{l}-\not{k})$$

we have

$$iq_\mu.I_1 = e^2\int\frac{d^4l}{(2\pi)^4}\mathrm{tr}\left[\gamma^5\frac{(\not{l}-\not{k})}{(l-k)^2}\gamma^\lambda\frac{\not{l}}{l^2}\gamma^\nu+\gamma^5\gamma^\lambda\frac{\not{l}}{l^2}\gamma^\nu\frac{(\not{l}+\not{p})}{(l+p)^2}\right]$$

now pass γ^ν through γ^5 in the second term and shift the integral over the first term according to

$$l\to(l+k)$$

leaving

$$iq_\mu.I_1 = e^2\int\frac{d^4l}{(2\pi)^4}\mathrm{tr}\left[\gamma^5\frac{\not{l}}{l^2}\gamma^\lambda\frac{(\not{l}+\not{k})}{(l+k)^2}\gamma^\nu-\gamma^5\frac{\not{l}}{l^2}\gamma^\nu\frac{(\not{l}+\not{p})}{(l+p)^2}\gamma^\lambda\right].$$

- This expression is manifestly antisymmetric under the interchange of (p, ν) and (k, λ), so the contribution from the second diagram cancels this.

However

- Because this derivation involves a shift of the integration variable, we have to be careful about whether this shift is allowed by the regularisation
- These integrals are divergent.
- If we use a momentum cut-off, or Pauli-Villars regularisation it turns out that the shift leaves over a finite, non-zero term.
- This is similar to the analysis of QED vacuum polarisation which is fixed by dimensional regularisation.
- For the axial vector current even dimensional regularisation contains a subtlety.
- It is due to the fact that γ^5 is intrinsically 4-dimensional. (In dim reg we use extra dimensions to regularise the integrals.)
- 't Hooft and Veltman in their original paper defined

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

  in d- dimensions.
- Consequence of this is that

$$\gamma^5 \,\text{anticommutes with}\, \gamma^\mu \,\text{for}\, \mu = 0, 1, 2, 3$$

  and

$$\gamma^5 \,\text{commutes with}\, \gamma^\mu \,\text{for}\, \mu = \text{others}$$

- In evaluating the integral in dim reg the momentum p, k and q all live in 4-d but the loop momentum l has components in all dimensions.
- Write

$$l = l_\parallel + l_\perp,$$

  where the first term has nonzero components in dimensions $0, 1, 2, 3$ and the second terms has nonzero components in the other $d - 4$ dimensions.

- Due to the commutation relations for γ^5 the identity for $q_\mu \gamma^\mu \gamma^5$ is now

$$q_\mu \gamma^\mu \gamma^5 = (\not{l} + \not{k})\gamma^5 + \gamma^5(\not{l} - \not{p}) - 2\gamma^5 \not{l}_\perp$$

- First two terms cancel as before, but the third term give the additional contribution

$$iq_\mu . I_1 = e^2 \int \frac{d^4 l}{(2\pi)^4} \text{tr}\left[-2\gamma^5 \not{l}_\perp \frac{(\not{l} - \not{k})}{(l-k)^2}\gamma^\lambda \frac{\not{l}}{l^2}\gamma^\nu \frac{(\not{l} + \not{p})}{(l+p)^2}\right]$$

- To evaluate this we combine denominators and shift the integration variables

$$l \to l + P$$

where

$$P = xk - yp$$

- After some manipulation we have to evaluate the integral

$$\int \frac{d^4 l}{(2\pi)^4} \frac{\not{l}_\perp \not{l}_\perp}{(l^2 - \Delta)^3}$$

where

$$\Delta = \Delta(k, p, x, y)$$

using

$$(\not{l}_\perp)^2 = l_\perp^2 \to \frac{(d-4)}{d} l^2$$

under the integration the integral can be written as

$$\frac{i}{(4\pi)^{d/2}} \frac{(d-4)}{2} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(3)\Delta^{2-d/2}} \underset{d \to 4}{\longrightarrow} \frac{-i}{2(4\pi)^2}$$

- Notice that the integral which seems to have a logarithmic divergence is finite using dim reg
- The remainder of the calculation is fairly straightforward. The terms involving the momentum shift P cancel and we find

$$\begin{aligned} iq_\mu . I_1 &= e^2 \left(\frac{-i}{2(4\pi)^2}\right) \text{tr}\left[2\gamma^5(-\not{k})\gamma^\lambda \not{p} \gamma^\nu\right] \\ &= \frac{e^2}{4\pi^2} \epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta \end{aligned}$$

- This is symmetric under interchange of (p,ν) and (k,λ) so the second diagram gives an identical contribution.

- Thus:

$$\begin{aligned}\langle p,k|\partial_\mu j^{\mu 5}(x)|0\rangle &= \frac{-e^2}{2\pi^2}\epsilon^{\alpha\lambda\beta\nu}(-ip_\alpha)\epsilon^*_\nu(p)(-ik_\beta)ep^*_\lambda(k) \\ &= \frac{-e^2}{2\pi^2}\langle p,k|\epsilon^{\alpha\lambda\beta\nu}F_{\alpha\nu}F_{\beta\lambda}(0)|0\rangle\end{aligned}$$

Which is the same as the ABJ anomaly equation.

10.2 Chiral Anomalies and Chiral Gauge Theories

- Up to now we have coupled gauge fields to fermions in a parity symmetric way
- That is we have replaced the derivative in the Dirac equation with a covariant derivative
- This means that the gauge fields couple to the vector current of fermions
- This only gives a subset of the possible couplings
- Now we will construct parity-asymmetric couplings and discuss their interplay with the axial current

First consider massless fermions

- No mass terms means no mixing between the LH and RH components of our Dirac fermions
- Thus in this theory we can write the kinetic energy terms in terms of Weyl Spinors

$$\mathcal{L} = \psi^\dagger_{Li} i\overline{\sigma}.\partial\psi_{Li} + \psi^\dagger_{Ri} i\sigma.\partial\psi_{Ri}$$

- We are free to couple this system to a gauge field by assigning the LH fields to one representation, say r of G and the RH to another.
- Let us say that they do not transform under G.
- Doing so we adapt the derivatives as normal to give

$$\mathcal{L} = \psi^\dagger_{Li} i\overline{\sigma}.D\psi_{Li} + \psi^\dagger_{Ri} i\sigma.\partial\psi_{Ri}$$

with $D_\mu = \partial_\mu - igA^a_\mu T^a_r$

- We can write this in terms of more traditional Dirac spinors as

$$\mathcal{L} = \overline{\psi}_i i\gamma^\mu \left(\partial_\mu - igA^a_\mu T^a_r \left(\frac{1-\gamma^5}{2}\right)\right)\psi_i$$

- The classical Lagrangian is invariant under

$$\psi \to \left(1 + i\alpha^a T^a_r \left(\frac{1-\gamma^5}{2}\right)\right)\psi$$

$$A^a_\mu \to A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c$$

- The concept of only LH fields coupling to gauge fields is familiar to us through the weak interactions

- A useful feature of chirally coupled fermions is that we can write RH fields as LH fields by writing

$$\sigma^2 \psi^*_R$$

which transforms as a LH field under the Lorentz transformations

- This allows us to write the entire Lagrangian in terms of "LH" fields.

- Define these new LH fields as

$$\psi'_L = \sigma^2 \psi^*_R, \qquad \psi'^\dagger_L = \psi^T_R \sigma^2$$

- This relabels the RH fermions as antifermions and calls their LH antiparticles a new species of LH fermion

- We can thus rewrite the Lagrangian for RH fields as (using integration by parts)

$$\int d^4x \psi^\dagger_{Ri} i\sigma.\partial\psi_{Ri} = \int d^4x \psi'^\dagger_{Li} i\overline{\sigma}.\partial\psi'_{Li}$$

- Notice that if these (originally RH) fermions transformed in rep r this manipulation changes the way they transform

- Their covariant derivative will now be

$$\begin{aligned}
\psi^\dagger_R i\sigma.(\partial - igA^a T^a_r)\psi_R &= \psi'^\dagger_L i\overline{\sigma}.(\partial + igA^a (T^a_r)^T)\psi'_L \\
&= \psi'^\dagger_L i\overline{\sigma}.(\partial - igA^a T^a_{\overline{r}})\psi'_L
\end{aligned}$$

Using

$$T^a_{\overline{r}} = -(T^a_r)^* = -(T^a_r)^T$$

- Thus the new fields transform as the conjugate representation to r.

Example: QCD

- n_f flavours of massless fermions
- This theory describes a gauge theory coupled to n_f massless fermions in the 3 and n_f flavours in the $\overline{3}$ representation of $SU(3)$
- The most general gauge theory of massless fermions would simply assign the LH fermions to an arbitrary reducible representation R of the gauge group G
- We have just seen that rewriting the system of dirac fermions leads to $R = r \oplus \overline{r}$ as real representation.
- A real representation is such that there exists a unitary transformation U

$$T^a_{\overline{R}} = U T^a_R U^\dagger$$

- Conversely if R is not a real representation, then the theory cannot be written in terms of Dirac fermions and is intrinsically chiral
- The mass terms for our QCD example are written as

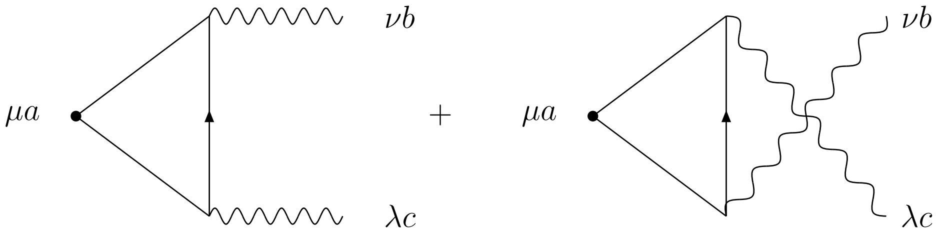
$$m\overline{\psi}_i\psi_i = m(\psi^\dagger_R\psi_L + \text{h.c.}) = -m(\psi'^T_{Li}\sigma^2\psi_{Li} + \text{h.c.})$$

- This has the form of a Majorana mass term.
- Given this the most general mass term that can be built purely from LH fermion fields is

$$\Delta\mathcal{L}_M = M_{ij}\psi^T_{Li}\sigma^2\psi_{Lj} + \text{h.c.}$$

  this term is symmetric under exchange of i and j.
- This term is gauge invariant under G.
- In general, we can always write down a mass term if the representation containing the fermions is strictly real.
- In an intrinsically chiral theory there is no possible invariant mass term
- At the classical level there is no restriction on the representation R of the LH fermions.

- However, at the loop level many choices are made inconsistent due to the axial vector anomaly
- Now consider computing the following diagrams in the context of a theory of massless LH fermions



- The vertex marked with a dot represents the gauge symmetry current

$$j^{\mu a} = \overline{\psi}\gamma^{\mu}\left(\frac{1-\gamma^5}{2}\right)T^a\psi$$

- There is also a factor of $(1-\gamma^5)/2$ at the gauge boson vertices
- Following the derivation for the ABJ anomaly above, the term containing a γ^5 has an axial anomaly that leads to the relation

$$\langle p, \nu, b; k, \lambda, c|\partial_{\mu}j^{\mu a}|0\rangle = \frac{g^2}{8\pi^2}\epsilon^{\alpha\nu\beta\lambda}p_{\alpha}k_{\beta}.\mathcal{A}^{abc},$$

where

$$\mathcal{A}^{abc} = \mathrm{tr}[T^a\{T^b, T^c\}]$$

and is a trace over group matrices in the representation R.
$\rightarrow$ this means that the current is conserved if A^{abc} vanishes

- If the current is not conserved it "does violence to the theory"
- For example: the gauge bosons receive divergent masses
the delicate relation between the three and four point interactions is disturbed
- We must therefore impose that the anomaly index, $A^{abc} = 0$ for chiral gauge theories

Example

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

- If the two gauge bosons in the anomaly diagrams are $SU(2)_L$ gauge bosons and the current is an $SU(2)_L$ gauge current then by using

$$T^a = \frac{\sigma^a}{2}$$

and

$$\{\sigma^b, \sigma^c\} = 2\delta^{bc}$$

this gives

$$\mathcal{A}^{abc} = \frac{1}{8}\mathrm{Tr}[\sigma^a 2\delta^{bc}] = 0$$

so this anomaly is zero

- If Q_L and L_L couple to the EM field there is an additional consistency condition found by the taking the current to be EM

$$\Rightarrow \mathcal{A}^{abc} = \mathrm{Tr}[Q\{T^b, T^c\}] = \frac{1}{2}\mathrm{Tr}(Q)\delta^{bc}$$

where Q is the matrix of electric charges of the fermions

- $\mathrm{Tr}(Q)$ is thus the sum of the fermion electric charges

- For our example we have (remembering the factor of 3 for colour)

$$\mathrm{Tr}(Q) = 3(\frac{2}{3} - \frac{1}{3}) + (0 - 1) = 0$$

- We see straight away that $SU(2)$ weak interactions can be combined with QED only if the theory contains equal numbers of quark and lepton doublets

Now more generally

- If the fermion representation R is real, R is equivalent to its conjugate representation $\overline{R}$.

- From previously we had that for real reps T_R is related by a unitary transformation to $T^a_{\overline{R}} = -(T^a_R)^T$.

- Since the anomaly coefficient is invariant under unitary transformations we can simply replace T_R by $T^a_{\overline{R}}$ then

$$\mathcal{A}^{abc} = \mathrm{Tr}[(-T^a)^T\{(-T^b)^T, (-T^c)^T\}] = -\mathrm{Tr}[\{T^b, T^c\}T^a] = -\mathcal{A}^{abc}$$

which means that the anomaly coefficient must be zero and therefore for any real representation the gauge theory will be automatically anomaly free

In general circumstances...

- We simplify the calculation of $\mathcal{A}$ by noting that it is an invariant of the gauge group G that is **totally symmetric** with three indices in the adjoint representation.
- For some possible groups a suitable invariant may not exist and in those cases $\mathcal{A}^{abc}$ vanishes.

Example: $SU(2)$

- The adjoint representation has spin 1
- The symmetric product of two spin 1 multiplets give a spin 0 and spin 2, with **no spin 1** component.
- This means that there is no symmetric tensor coupling two spin 1 indices to give a spin 1
- $\mathcal{A}^{abc}$ in this case must therefore vanish in any $SU(2)$ gauge theory - as we showed explicitly above

$SU(N)$ **with** $N \geq 3$

- There is a unique symmetric invariant d^{abc} of the required type
- It appears in the anti-commutator of the representation matrices of the fundamental representation

$$\{T_F^a, T_F^b\} = \frac{1}{n}\delta^{ab} + d^{abc}T_F^c$$

- The uniqueness of this invariant implies that, in an $SU(N)$ gauge theory,any form of the trace of the form $\mathcal{A}^{abc} = \mathrm{tr}[T^a\{T^b, T^c\}]$ is proportional to d^{abc}.
- For each representation r, we can define an **anomaly coefficient** $A(r)$ by

$$\mathrm{Tr}[T_r^a\{T_r^b, T_r^c\}] = \frac{1}{2}A(r)d^{abc}$$

- For the fundamental representation, it is clear that

$$A(F) = 1.$$

- It follows also that

$$A(\overline{r}) = -A(r)$$

- For higher representations, the anomaly coefficients can be calculated. For example the result for the antisymmetric (a) and symmetric (s) two-index representation of $SU(N)$,

$$\text{Dimension}(a) = \frac{N(N-1)}{2} \qquad A(a) = N-4$$
$$\text{Dimension}(s) = \frac{N(N+1)}{2} \qquad A(s) = N+4$$

Other simple Lie groups

- Only $SU(N)$, $SO(4N+2)$ and E_6 have complex representations (for purely real we remember that there are no anomalies).
- Of these only $SU(N)$ and $SO(6)$, (which has the same algebra as $SU(4)$), have the symmetric invariant of the type required to form the anomaly
- This means that gauge theories based on $SO(4N+2)$, for $N \geq 2$ and on E_6 are automatically invariant
- There is one more constraint on the representation content of a chiral gauge theory, which come from considering its coupling to gravity.
- It is possible to show that the triangle diagrams above give an anomaly contribution when computed with a gauge current and external gravitational fields.
- The group theory factor that multiplies this diagram is

$$\text{Tr}(T^a_r)$$

- This factor vanishes automatically for non-abelian symmetries, only abelian symmetries can give a non-zero contribution.

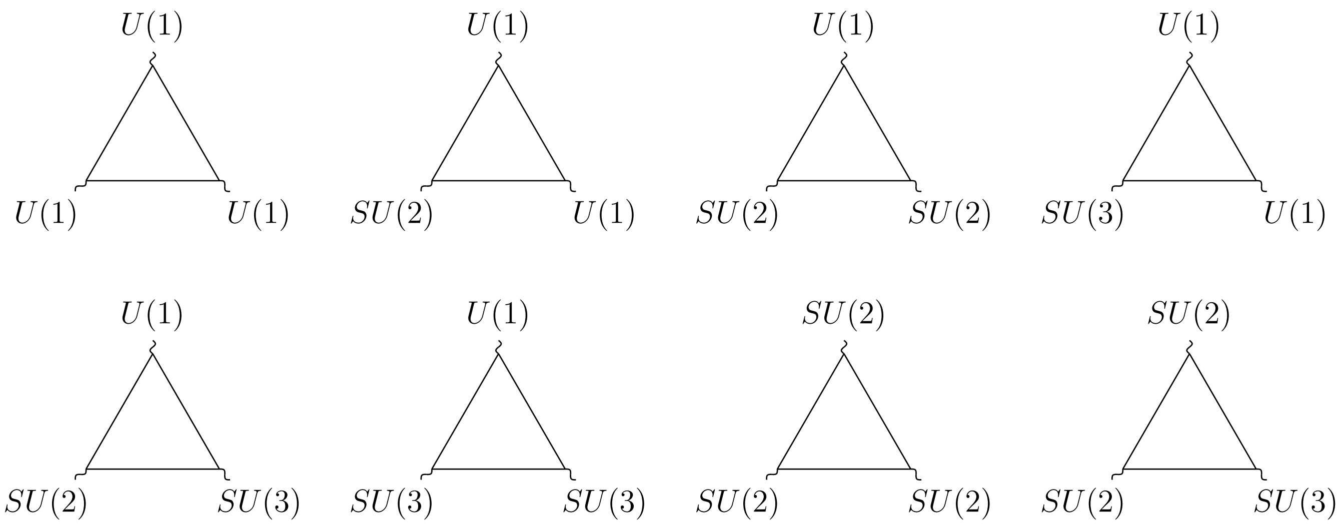
10.3 Cancellation of anomalies in the Standard Model.

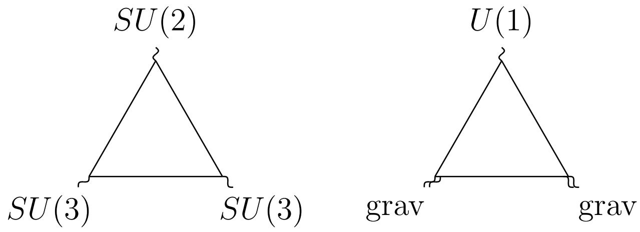
- We know that theories in which gauge bosons couple to chiral currents can be gauge invariant only if the anomalous contributions disappear.
- GWS (Glashow-Weinberg-Salam) theory is a chiral, weak interaction current are left handed.
- We must check that the anomalous terms from the triangle diagrams cancel.

- Anomalous terms of the triangle diagrams of 3 gauge bosons are proportional to

$$\text{Tr}[\gamma^5 T^a \{T^b, T^c\}]$$

- The γ^5 registers the fact that the anomaly is associated with chiral currents.
- This factor gives a $+1$ for RH fermions and -1 for LH fermions
- In theories of QCD and QED, in which the gauge bosons couple equally to RH and LH species, they automatically cancel.
- To evaluate the anomalies of the GWS theory, it is easiest to work with the $SU(2)_L \times U(1)_Y$ gauge bosons.
- It is sufficient to evaluate the triangle diagrams for massless fermions, that way we can assign distinct quantum numbers to LH and RH fermions.
- We must consider not only the anomalies of diagrams with four $SU(2) \times U(1)_Y$ gauge bosons, but also diagrams with weak interactions and colour $SU(3)_c$ gauge bosons of QCD
- We should also consider possible anomalies coming from a weak-interaction gauge boson and two gravitions
- We can omit diagrams, such as the anomaly of 3 $SU(3)_c$ bosons or of one $SU(3)_c$ boson and two gravitons, in which all of the couplings are LR symmetric.
- The full set of diagrams with possible anomalous terms are:





- The $SU(2)^3$ anomaly is zero as we have shown previously.
- The diagrams with one $SU(2)$ or $SU(3)$ are proportional to

$$\mathrm{Tr}[T^a] = 0, \qquad \mathrm{Tr}[\tau^a] = 0$$

  where T^a and τ^a we use to label the $SU(3)$ and $SU(2)$ generators.
- The remaining anomalies are non-trivial

  a) One $U(1)$ with two $SU(3)$

  b) One $U(1)$ with two $SU(2)$

  c) Three $U(1)$s

  d) One $U(1)$s with two gravitons
- Look at a). Anomaly of one $U(1)$ with two $SU(3)$ is proportional to

$$\mathrm{Tr}[T_F^a T_F^b Y] = \frac{1}{2}.\delta^{ab} \sum_q Y_q$$

  where the sum runs over LH quarks and RH quarks, with an extra (-1) for the LH contributions.
- We have the particle assignments in Table 2
- Inserting the charge assignments for u_L, u_R, d_L and d_R we have

$$\sum_q Y_q = 3 \times \left[-\frac{1}{6} - \frac{1}{6} + \frac{2}{3} - \frac{1}{3} \right] = 0$$

- Similarly, for the anomaly b), one $U(1)$ with two $SU(2)$, we find

$$\mathrm{Tr}[\tau^a \tau^b Y] = \frac{1}{2}\delta^{ab} \sum_{f_L} Y_{f_L}$$

Fermion (3 families)	$SU(3)_C$, $SU(2)_L$, $U(1)_Y$
$Q = (u_L \;\; d_L)$	$(\mathbf{3},\ \mathbf{2}\ ,\ \frac{1}{6})$
u_R	$(\mathbf{3},\ \mathbf{1},\ \frac{2}{3})$
d_R	$(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3})$
$L = (\nu_L \;\; e_L)$	$(\mathbf{1},\ \mathbf{2}\ ,\ -\frac{1}{2})$
e_R	$(\mathbf{1},\ \mathbf{1},\ -1)$

Table 2: Standard Model particle assignments

where f_L runs over the LH fermions

$$\sum_{f_L} Y_{f_L} = 3\left[-\left(-\frac{1}{2}\right) - 3.\frac{1}{6} = 0\right]$$

factor of 3 takes account of colour.

- c) the anomaly of 3 $U(1)$s is proportional to

$$\mathrm{Tr}[Y^3] = 3\left[-2\left(-\frac{1}{2}\right)^3 + (-1)^3 + 3\left[-2\left(\frac{1}{6}\right)^3 + \left(\frac{2}{3}\right)^3 + \left(-\frac{1}{3}\right)^3\right]\right]$$

- Finally, d) the gravitational anomaly with one $U(1)$ is proportional to

$$\mathrm{Tr}[Y] = 3\left[-2\left(-\frac{1}{2}\right) + (-1) + 3\left[-2\left(\frac{1}{6}\right) + \left(\frac{2}{3}\right) + \left(-\frac{1}{3}\right)\right]\right]$$

- We thus find that the GWS theory is chiral gauge theory with no axial vector anomalies.
- The consistency of the theory requires that quarks and leptons appear in equal numbers, organising themselves into successive generations in this way.

11 Chiral Lagrangian approximation

- We know α_{QCD} becomes large at small energies.
- Do not expect that QCD can be described in terms of quarks and gluons
- The "confinement hypothesis" asserts that only objects which are colour singlets may be identified with free particles
- As we have seen before it is easy to construct combinations of quarks into colour singlets

$$q\bar{q} : \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$$

  and the totally antisymmetric combination of three quarks

$$qqq : (\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3})_A = \mathbf{1}$$

- QCD also has colourless combinations of gluons - glue-balls
- Symmetric combinations of gluons

$$(\mathbf{8} \otimes \mathbf{8})_S = \mathbf{1} \oplus ...$$

  and

$$(\mathbf{8} \otimes \mathbf{8} \otimes \mathbf{8})_S = \mathbf{1} \oplus ...$$

- These states are hard to find as they do not have electromagnetic interactions
- Mesons and Baryons are not of course as simple as this they are a mess of quark-antiquark pairs and glue i.e. schematically

$$\text{mesons} : \qquad |q^i \bar{q}^j (1 + q^k \bar{q}^k + \text{glue} + ...)\rangle, \tag{11.1}$$

$$\text{baryons} : \qquad |q^i q^j q^k (1 + q^l \bar{q}^l + \text{glue} + ...)\rangle, \tag{11.2}$$

- i, j, k, l are flavour indices, colour indices not shown but contracted in appropriate way to give colour singlet.
- Should be noted that any colourless combinations is actually a linear superposition of all possible colour singlets with the same quantum numbers
- Although the confinement hypothesis is consistent with experiment it is yet to be proven from first principles

- Solution lies in developments in non-perturbative field theory

How best can we relate QCD to the "nuclear force"? E.g. in $\pi - N$ scattering?

- The key is to understand that the electroweak quantum numbers shared by quarks (and their composites) allow us to see that the QCD lagrangian is spontaneously broken (presumably by the QCD force itself) whilst leaving QCD unbroken
- Opens up a way of describing the static limit of QCD

First let us be clear about what we are describing.

- QCD is the fundamental theory of strong interactions
- The two phases:

  quark-gluon phase at short distances
  baryon-meson phase at long distances

  are different phases of the same theory
- The intrinsic scale associated with QCD, Λ_{QCD} provides a quantitative way to differentiate between the two phases
- E.g. quark masses can be put into two categories:

$$m_q \gg \Lambda_{QCD} \text{ - quark can be viewed as quasi-free}$$
$$m_q \ll \Lambda_{QCD} \text{ - quark is strongly interacting}$$

- Two phases share some symmetries (e.g. charge invariance) others can be treated differently
- In **quark-gluon plasma** some symmetries are **linearly realised** where as in the **baryon-meson phase** are realised **non-linearly** (more later)
- Although we do not have the tools to derive the meson-baryon phase from first principals we can use the symmetries to gain valuable information

Setting up the analysis:

- Turn off electroweak interactions
- Consider QCD interactions with non-zero quark masses
- Segregate quark masses according to relative sizes to Λ_{QCD}:

$$\begin{aligned} m_u, m_d &\ll \Lambda_{QCD} \\ m_s &\sim \Lambda_{QCD} \\ m_c, m_b, m_t &\gg \Lambda_{QCD} \end{aligned}$$

  with c, b and t playing no part in the meson-baryon phases, i.e. large distance description of QCD.
- To start set $m_u = m_d = m_s = 0 \Rightarrow$ **the chiral limit**
- QCD lagrangian is then (in terms of the Dirac $\boldsymbol{q}$ field)

$$\mathcal{L}_{\mathcal{QCD}} = -\frac{1}{4}\mathrm{Tr}(\boldsymbol{G}_{\mu\nu}\boldsymbol{G}^{\mu\nu}) + \sum_{i=1}^{3} \overline{\boldsymbol{q}}_i \gamma_\mu D^\mu \boldsymbol{q}_i \tag{11.3}$$

- By setting the masses to zero, we have gained a very large global symmetry:

$$SU(3)_L \times SU(3)_R \times U(1)$$

- This global symmetry is linearly realised on the three light quarks, u, d, s

$$SU(3)_L : \boldsymbol{q}_L \to U_L \boldsymbol{q}_L; \qquad SU(3)_R : \boldsymbol{q}_R \to U_R \boldsymbol{q}_R;$$

$$U(1) : \boldsymbol{q}_{L,R} \to e^{i\alpha} \boldsymbol{q}_{L,R}$$

  this $U(1)$ is actually baryon number
- There is also a flavour blind chiral $U(1)$

$$\boldsymbol{q}_L \to e^{i\beta} \boldsymbol{q}_L; \qquad \boldsymbol{q}_R \to e^{-i\beta} \boldsymbol{q}_R$$

  but this symmetry is not conserved due to quantum effects (an anomaly) see later
- Above is in the quark phase, what happens in the baryon-meson phase
  - Lorentz invariance is unbroken
  - Colour invariance is unbroken
  - Chiral symmetries of massless QCD are spontaneously broken

– Vector symmetries of massless QCD is unbroken

- These are all assumptions but the first two can be argued to be justifiable through experimental observations
- The second two have important consequences
- If true the chiral symmetry is broken to its maximal vectorial subgroup

$$SU(3)_L \times SU(3)_R \to SU(3)_v$$

leaving the $U(1)$ unaffected.

What is responsible for the spontaneous symmetry breaking?

- Believed to take place through the condensation of quark pairs in the vacuum
- We don't actually need to know how this happens as we ca infer consequences via symmetry considerations
- Consider the fermion bi-linear of a quark and an anti-quark:

$$\boldsymbol{q}^{\dagger ia}_{L\alpha}\boldsymbol{q}^{\beta}_{Rjb}$$

where i and j are the $SU(3)_L$ and $SU(3)_R$ indices, α, β Lorentz indices, a, b are colour indices

- Under Lorentz the bilinear has the structure

$$(\mathbf{2},\mathbf{1}) \otimes (\mathbf{2},\mathbf{1}) \sim (\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1})$$

- Singlet combination means we do not break Lorentz invariance
- Similarly, under the colour group we have

$$\overline{\mathbf{3}^c} \otimes \mathbf{3}^c = \mathbf{1}^c \oplus \mathbf{8}^c$$

- Presence of a singlet means that this particular combination does not break colour.

What about the global symmetries?

- The $SU(3)_L \times SU(3)_R$ the bilinear transforms as $(\mathbf{3},\overline{\mathbf{3}})$

- This representations breaks up in terms of the diagonal vector-like subgroup $SU(3)_v$ as

$$(\mathbf{3}, \overline{\mathbf{3}}) \to \mathbf{1} \oplus \mathbf{8};$$

showing one component that breaks the chiral symmetry while preserving the vectorial subgroup

- It then follows that the bilinear achieves the desired effect:
The condensation of the bilinear breaks the chiral symmetry to the vectorial symmetry whilst preserving Lorentz and colour symmetries

So what we assume is the following...

- The dynamical assumption is simply that the effect of the QCD force between quarks and anti-quarks is to force $q\overline{q}$ condensates to form in the vacuum

$$\langle \boldsymbol{q}_L^{\dagger i} \boldsymbol{q}_{Rj} \rangle \neq 0$$

- This leaves the Lorentz and colour symmetries untouched but spontaneously breaks the chiral symmetries down to their diagonal subgroup

Counting the degrees of freedom

- Both $SU(3)_L$ and $SU(3)_L \times SU(3)_L$ have 8 generators each
- The spontaneous breakdown of

$$SU(3)_L \times SU(3)_R \to SU(3)_v$$

breaks 8 generators

- To each of the generators is associated a massless Nambu-Goldstone boson
- Each with exactly the same quantum numbers and which transform as an **octet under the** $SU(3)_v$

Note

- Although the full chiral symmetry is broken in the vacuum it is still non-linearly realised on the Nambu-Goldstone bosons as apposed to linearly realised on the quarks.

- This can be seen in the way the NGbs couple to matter
- In this realisation the fundamental construct which contains the NGbs is the group element which labels the broken symmetry operations
- We can write it to be the $n \times n$ complex matrix

$$\mathbf{\Sigma}(x) \equiv e^{\frac{2i}{F_\pi}\boldsymbol{\pi}(x)}$$

where F_π is a constant and

$$\boldsymbol{\pi}(x) = \boldsymbol{\lambda}^a \pi^a(x)$$

where $\pi^a(x)$ are the 8 NG fields and $\boldsymbol{\lambda}^a$ are the generators of $SU(3)$ normalised to

$$\mathrm{Tr}(\boldsymbol{\lambda}^a \boldsymbol{\lambda}^b) = 2\delta^{ab}$$

How does this object transform

- This object transforms in the same way under $SU(3)_L \times SU(3)_R$ as the quark condensate, i.e.

$$\mathbf{\Sigma}(x) \to \mathbf{\Sigma}(x)' = U_L \mathbf{\Sigma}(x) U_R^{-1}$$

where $U_{L,R}$ are unitary matrices generating $SU(3)_L$ and $SU(3)_R$

$$U_L = e^{\frac{i}{2}\boldsymbol{\omega}_L}; \qquad U_R = e^{\frac{i}{2}\boldsymbol{\omega}_R}$$

where

$$\boldsymbol{\omega}_{L,R} = \omega^a_{L,R}\boldsymbol{\lambda}_a.$$

- Under vectorial transformations $\boldsymbol{\omega}_L = \boldsymbol{\omega}_L \equiv \boldsymbol{\omega}_v, U_L = U_R$ and the NG fields transform linearly as

$$\delta\boldsymbol{\pi}(x) = \frac{i}{2}\left[\boldsymbol{\omega}_v, \boldsymbol{\pi}(x)\right]$$

where we have taken infinitesimal transformations.

- Under a chiral transformation the NG fields transform inhomogenously for which we take $\boldsymbol{\omega}_L = -\boldsymbol{\omega}_L \equiv \boldsymbol{\omega}_A, U_L = U_R$

$$\delta\boldsymbol{\pi}(x) = \frac{F_\pi}{2}\boldsymbol{\omega}_A - \frac{i}{2}\left\{\boldsymbol{\pi}, \boldsymbol{\omega}_A\right\} + ...$$

- This is a characteristic transformation law for NG bosons with the non-linear term which signifies spontaneous symmetry breaking

- The $\boldsymbol{\omega}_A^a$ label the degeneracy of the vacuum

What can we infer from this?

- 1) Mass terms which are quadratic in the pion fields are not invariant and thus **pions are massless.**
- 2) Pions must couple to matter in such a way that the Lagrangian is left invariant when the pion fields are shifted

$\rightarrow$ This means the lagrangian density may change at most by a total derivative
$\rightarrow$ **Pions must couple to terms which are themselves total derivatives**
(decoupling at zero momentum)

- Pion dynamics are described in terms of an effective lagrangian, invariant under $SU(3)_L \times SU(3)_R$ expressed in terms of $\boldsymbol{\Sigma}(x)$.
- Unique term involving least number of derivatives:

$$\mathcal{L}_0 = \frac{F_\pi^2}{8}\mathrm{Tr}(\partial_\mu \boldsymbol{\Sigma} \partial^\mu \boldsymbol{\Sigma}^{-1}). \tag{11.4}$$

- The dimensionful constant F_π is included to make the dimensions come out correctly
- Expanding in terms of F_π

$$\boldsymbol{\Sigma} = 1 + \frac{2i}{F_\pi}\boldsymbol{\lambda}^a \pi^a + ...,$$

  we obtain the kinetic term

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \boldsymbol{\pi}^a \partial^\mu \boldsymbol{\pi}^a) + ... \tag{11.5}$$

- together with other interaction terms all suppressed by $1/F_\pi$ - only a suppression for momentum less than F_π

Lagrangian is valid only when momentum less than F_π It is an effective theory designed to describe low energy behaviour - it is not renormalisable but this does not matter as we are not concerned with the high energy behaviour

What are these π fields?

- We can write them explicitly (putting in the forms of the $\boldsymbol{\lambda}$s in the following way...

$$
\begin{aligned}
\boldsymbol{\pi} &= \begin{bmatrix} \pi_3 + \frac{1}{\sqrt{3}}\pi_8 & \pi_1 - i\pi_2 & \pi_4 - i\pi_5 \\ \pi_1 + i\pi_2 & -\pi_3 + \frac{1}{\sqrt{3}}\pi_8 & \pi_6 - i\pi_7 \\ \pi_4 + i\pi_5 & \pi_6 + i\pi_7 & -\frac{2}{\sqrt{3}}\pi_8 \end{bmatrix} \\
&= \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}}\pi_3 + \frac{1}{\sqrt{6}}\pi_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi_3 + \frac{1}{\sqrt{6}}\pi_8 & K^0 \\ K^- & \overline{K}^0 & -\sqrt{\frac{2}{3}}\pi_8 \end{bmatrix}
\end{aligned}
$$

- The superscripts refer to the anticipated electric charge once we include electromagnetic interactions

- The higher order interaction terms in the lagrangian (11.5) give the interactions of multi NG bosons, giving $\pi - \pi$, $\pi - K$, $K - K$,... scatterings

11.1 Explicit Chiral Symmetry Breaking

- We know that quarks have masses and their effects must be incorporated into the Lagrangian

- Introducing light quark masses **Explicitly breaks the chiral symmetry** generating masses for the NG bosons

- To establish how this explicit breaking effects the chiral lagrangian
  $\rightarrow$ analyse the transformation properties of the masses under the chiral symmetry

- First observe: Simplest potential term we can write down is

$$\mathrm{Tr}(\boldsymbol{\Sigma})$$

  which is invariant under the vector subgroup.

- So we could add this multiplied by a universal mass term, giving masses to the quarks whilst retaining the vector global symmetry.

- **However** we know u, d, s have different masses so we are forced to introduce the light quark mass matrix as

$$\boldsymbol{M} \equiv \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

and we write

$$\mathcal{L}_{SB} = \frac{F_\pi^2}{8} m_0 \mathrm{Tr}(\boldsymbol{\Sigma}^\dagger \boldsymbol{M}) + \mathrm{c.c}$$

where m_0 is an unknown parameter with mass dimension 1.

- Looking at the quadratic terms in the NG bosons it is easy to find the following masses

$$\begin{aligned} m_{\pi^\pm}^2 &= m_0(m_u + m_d), \\ m_{K^\pm}^2 &= m_0(m_u + m_s), \\ m_{K^0}^2 &= m_0(m_d + m_s), \end{aligned}$$

- The diagonal fields appear as

$$m_0 \left((m_u + m_d)\pi^3\pi^3 + \frac{2}{\sqrt{3}}(m_u - m_d)\pi^3\pi^8 + \frac{1}{3}(m_u + m_d + 4m_s)\pi^8\pi^8 \right)$$

- We can diagonalise this to mass eigenstates π^0 and η with masses

$$\begin{aligned} m_{\pi^0}^2 &= m_0 \left(m_u + m_d - \frac{1}{2}\frac{(m_u - m_d)^2}{2m_s - m_u - m_d} \right) \\ m_{\eta^0}^2 &= m_0 \left(\frac{m_u + m_d + 4m_s}{3} + \frac{1}{2}\frac{(m_u - m_d)^2}{2m_s - m_u - m_d} \right) \end{aligned}$$

- Although of the correct sign the extra term does not account for the π^+ and π^+ mass difference $\rightarrow$ need emag interactions

- Despite this we can use the expression to write

$$m_\eta^2 + m_{\pi^0}^2 = \frac{2}{3}(m_{K^0}^2 + m_{K^\pm}^2 + m_{\pi^\pm}^2)$$

which is in agreement with experiment apart from corrections due to emag contributions.

- Expanding

$$\mathcal{L}_{SB} = \frac{F_\pi^2}{8} m_0 \mathrm{Tr}(\boldsymbol{\Sigma}^\dagger \boldsymbol{M}) + \mathrm{c.c}$$

further we get higher order terms which give corrections to the scattering of the NG bosons $\Rightarrow$ **Whole effective Lagrangian provides a model for the interactions of π, K, η.**

11.2 Electroweak Interactions

We now need to add electroweak interactions

- W, Z and γ couple to currents in quark and lepton fields, corresponding to symmetries which appear in the chiral lagrangian.
- Proceed as before, write the NG bosons in terms of $\mathbf{\Sigma}$ and then couple to gauge bosons
- Canonical $SU(3)_L \times SU(3)_R$ currents can be obtained using Noether process:

$$J^{\mu a}_{L,R} = \frac{\delta \mathcal{L}_0}{\delta \partial_\mu \pi^b} \frac{\delta \pi^b}{\delta \omega^a_{L,R}}$$

- Complicated expression, expand in powers of F_π^{-1}

$$J^{\mu a}_{L,R} = \pm \frac{F_\pi}{4} \partial_\mu \pi^a(x) + ...$$

  Compare this with

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \boldsymbol{\pi}^a \partial^\mu \boldsymbol{\pi}^a) + ...$$

  we see that we can write the lagrangian as

$$\mathcal{L}_0 = -\frac{1}{F_\pi} \pi^a \partial^\mu (J^a_{\mu L} - J^a_{\mu R})) + ...$$

- Exactly the coupling a NG boson should have to the currents associated with the spontaneously broken symmetries

What about a full expression for the currents?

- For a better but longer derivation see Georgi - Weak interactions see his website
- We will use symmetry arguments:
- Left current: Built out of $\mathbf{\Sigma}$ and transforms as a Lorentz 4-vector, an octet under $SU(3)_L$ and a singlet under $SU(3)_R$ $\Rightarrow$ must thus include $\partial \mathbf{\Sigma}$ or $\partial \mathbf{\Sigma}^{-1}$
  $\Rightarrow$ Include $\boldsymbol{\lambda}^a$ giving the correct $SU(3)$ transformation with

$$\boldsymbol{\lambda}^a \rightarrow U^{-1}_{L,R} \boldsymbol{\lambda}^a U_{L,R}$$

  under the chiral symmetry.
  THUS: the following transforms as an $SU(3)_L$ current

$$\mathrm{Tr}(\mathbf{\Sigma}^{-1} \boldsymbol{\lambda}^a \partial_\mu \mathbf{\Sigma})$$

- Making up the dimensions with powers of F_π the two current are

$$J^a_{\mu L} = i\frac{F_\pi^2}{2}\mathbf{Tr}(\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}^a\partial_\mu\mathbf{\Sigma})$$

$$J^a_{\mu R} = i\frac{F_\pi^2}{2}\mathbf{Tr}(\mathbf{\Sigma}\boldsymbol{\lambda}^a\partial_\mu\mathbf{\Sigma}^{-1})$$

expanding in inverse powers of F_πs we have

$$J^a_{\mu L,R} = \pm F_\pi \mathrm{Tr}(\boldsymbol{\lambda}^a\partial_\mu\boldsymbol{\pi}) - i\mathrm{Tr}([\boldsymbol{\pi},\boldsymbol{\lambda}^a]\,\partial_\mu\boldsymbol{\pi}) \mp \frac{2}{3F_\pi}\mathrm{Tr}([\boldsymbol{\pi},[\boldsymbol{\pi},\boldsymbol{\lambda}^a]]\,\partial_\mu\boldsymbol{\pi}) + ...$$

- Using these currents we can build the currents that couple to the gauge bosons

Example - Photon

- The photon couples to the three light quarks with strength equal to their charges
- Thus the photon coupling is determined by demanding invariance under local vector-like transformations of the form

$$\mathbf{\Sigma} \to e^{ial(x)\boldsymbol{Q}}\mathbf{\Sigma}e^{-ial(x)\boldsymbol{Q}}, \qquad \delta\mathbf{\Sigma} = i\alpha(x)\,[\boldsymbol{Q},\mathbf{\Sigma}]\,,$$

where

$$\boldsymbol{Q} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} = \frac{1}{2}\boldsymbol{\lambda}_3 + \frac{1}{2\sqrt{3}}\boldsymbol{\lambda}_8$$

- In order to ensure the lagrangian is gauge invariant we replace the ∂_μ by the covariant derivative

$$D_\mu = \partial_\mu\mathbf{\Sigma} + ieA_\mu\,[\boldsymbol{Q},\mathbf{\Sigma}]$$

- It is now easy to see that the emag current is

$$J^{\mathrm{em}}_\mu = J^3_{\mu L} + \frac{1}{\sqrt{3}}J^8_{\mu L} + (L \to R).$$

- Clearly the emag interactions are not invariant under all chiral symmetries
- Chiral symmetries corresponding to the charged mesons ($\pi^\pm$ and $K^\pm$) are broken equally

Electromagnetic contributions to meson masses

- One immediate effect of this is that these mesons will get contributions to their masses from the emag interaction

- Parameterise these contributions by Δm^2 giving

$$
\begin{aligned}
m^2_{\pi^\pm} &= m_0(m_u + m_d) + \Delta m^2, \\
m^2_{K^\pm} &= m_0(m_u + m_s) + \Delta m^2, \\
m^2_{K^0} &= m_0(m_d + m_s) \\
m^2_{\pi^0} &= m_0 \left(m_u + m_d - \frac{1}{2} \frac{(m_u - m_d)^2}{2m_s - m_u - m_d} \right) \approx m_0 \left(m_u + m_d \right) \\
m^2_{\eta^0} &= m_0 \left(\frac{m_u + m_d + 4m_s}{3} + \frac{1}{2} \frac{(m_u - m_d)^2}{2m_s - m_u - m_d} \right) \approx m_0 \left(\frac{m_u + m_d + 4m_s}{3} \right)
\end{aligned}
$$

- **Now we need measurements of the masses to fit the parameters**

- Fitting the parameters to the π and K masses:

$$
\begin{aligned}
m_0 m_u &= 0.00649 \text{ GeV}^2 \qquad m_0 m_d = 0.01172 \text{ GeV}^2 \\
m_0 m_s &= 0.23596 \text{ GeV}^2 \qquad \Delta m^2 = 1232 \text{ GeV}^2
\end{aligned}
$$

  yielding

$$
\frac{m_u}{m_d} \approx \frac{1}{2}, \qquad \frac{m_d}{m_s} \approx \frac{1}{20}
$$

  and a prediction for the η which is good to around 3%.

- See Georgi/Ramond + many others for more details.

One last thing... Anomalies of Chiral Currents

- Up to now we have discussed the chiral symmetries of QCD according classical current conservation.

- We must ask how everything is affected by the ABJ anomaly, and what consequences these modifications have.

- For the axial currents of QCD,

$$
j^{\mu 5} = \overline{\psi} \gamma^\mu \gamma^5 \psi \qquad j^{\mu 5 a} = \overline{\psi} \gamma^\mu \gamma^5 \tau^a \psi,
$$

  we can read off the group theory factors for the ABJ anomaly (see section on anomalies).

- For the axial isospin currents,

$$\partial_\mu j^{\mu a} = -\frac{g^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F^c_{\alpha\beta}F^d_{\mu\nu}\mathbf{Tr}[\tau^a T^c T^d],$$

  where $F^d_{\mu\nu}$ is a gluon field strength, τ^a is an isospin matrix, T^c is a colour matrix and the trace is take over all colours and flavours.

- We know of course that this contribution is vanishing as

$$\mathrm{Tr}[\tau^a T^c T^d] = \mathrm{Tr}[\tau^a]\mathrm{Tr}[T^c T^d],$$

  since the trace of a single τ vanishes.

- Axial isospin current is unaffected by the ABJ anomaly

- However, the isospin singlet axial current, the τ is replaced by a 1 on flavours, and we find

$$\partial_\mu j^{\mu 5} = -\frac{g^2 n_f}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F^c_{\alpha\beta}F^c_{\mu\nu}.$$

- Therefore isospin singlet axial current is not conserved.

- This explains why there is no light isosinglet pseudoscalar with mass comparable to that of the pions

- Though the axial isospin current have no axial anomaly from QCD, they do have an anomaly associated with the coupling of quarks to emag.

- The emag anomaly of the axial isospin currents is given by

$$\partial_\mu j^{\mu 5a} = -\frac{e^2 n_f}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\mathrm{Tr}[\tau^a Q^2],$$

  where now $F_{\mu\nu}$ is the emag field strength and

$$Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}.$$

- The flavour trace is only non-zero for $a = 3$; in that case the anomaly is

$$\partial_\mu j^{\mu 53} = -\frac{e^2 n_f}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$$

- Because the current $j^{\mu 53}$ annihilated a π^0 meson, the above indicates that the anomaly contributes to the matrix element for the decay

$$\pi^0 \to \gamma\gamma$$

- Doing the computation (see Peskin and Schroeder (P672-676) or Georgi's book on weak interactions) we find

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2}{64\pi^3} \frac{m_\pi^3}{f_\pi^2}$$