Description and Evaluation of Josephson Effect Derived Qubits using Superfluid Helium-4

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Abstract

An equivalent circuit theory used to describe and quantize superconducting circuits was extended to describe superfluid helium devices. Classical hydrodynamics was combined with the Landau model of superfluids to obtain a classical circuit theory. This was adapted to the 'branch variable' language used to describe superconducting quantum circuits. The resulting theory was used to derive equations of motion for a hypothetical superfluid analogue to the superconducting charge qubit. A quantum Hamiltonian of the system was derived, and found to be identical to that of the charge qubit – however, the expected regime of operation was found to be physically unreasonable at typical scales of microfluidic components.

Basic Equations

Properties of superfluid helium-4

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p \qquad \boldsymbol{u} = \frac{\hbar}{m}\nabla\phi \qquad p = \frac{\rho}{m_4}\mu$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} = -\frac{\mu}{\hbar}$$

Classical Kirchhoff variables

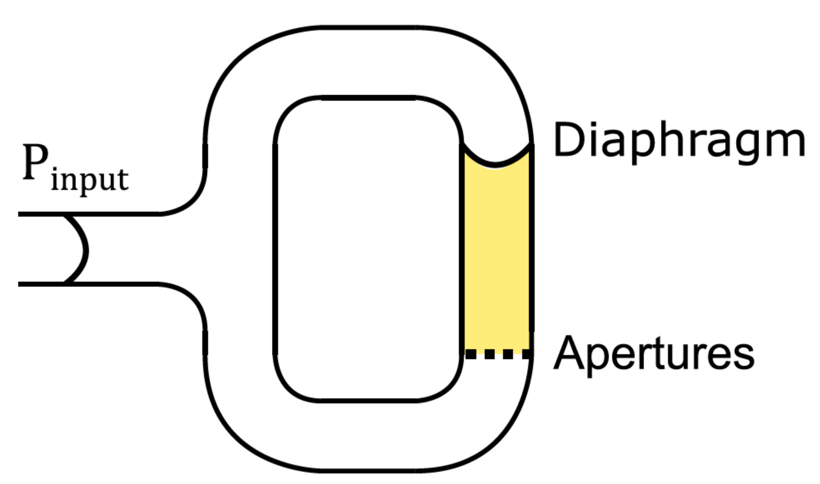
pressure difference Δp mass current $I = \frac{dm}{dt}$

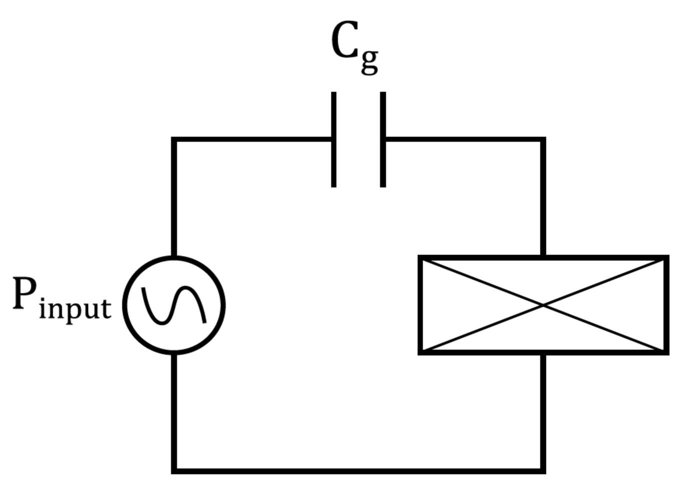
New Kirchhoff (branch) variables

'flux' $\Phi = \int \mu(t')dt'$ particle number $N = \frac{1}{m}\int I(t')dt'$

Enables construction of Lagrangians and Hamiltonains by application of a set of circuit analysis rules [1].

The Qubit System

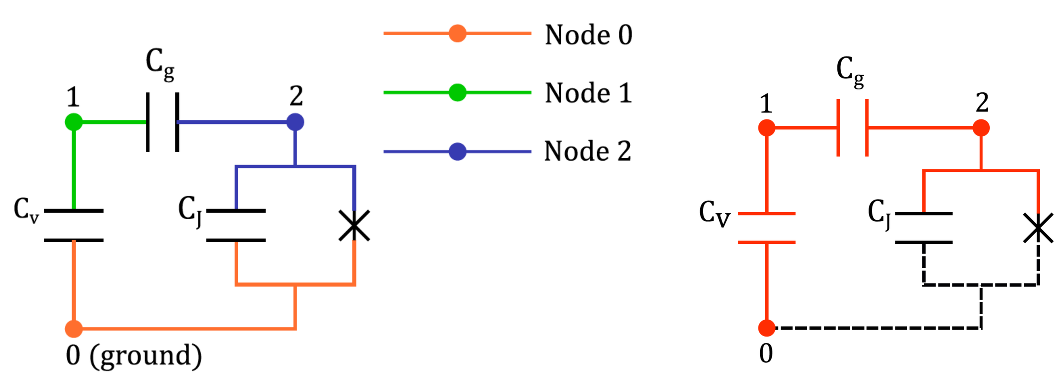




Physical appearance of hypothetical qubit system. A small volume of superfluid (yellow) is confined by a diaphragm on one end and an array of nanoapertures (Josephson junctions) on the other. An external pressure is applied to the circuit, represented by another diaphragm, which could be electromechanically driven. In the qubit regime, helium-4 atoms may tunnel through the weak link so that there is a meaningful energy change in storing even one additional atom in the reservoir, creating a two-state system

Equivalent hydrodynamic circuit modelling the system. The internal diaphragm is modelled by a 'gate' capacitance, while the weak link array is modelled as a generic Josephson element (the 'cross-box' symbol), which may include parallel conductance or capacitance. This circuit is identical to that of the superconducting charge qubit [2].

Circuit Analysis



A parallel-capacitance model is chosen for the Josephson element, as the extreme low-temperature regime of operation of the qubit allows exclusion of any residual resistive effects due to normal fluid. The circuit is separated into a 'spanning tree' and fluxes Φ_i are assigned to each node. The classical lagrangian of the circuit may be obtained

$$\mathcal{L} = \frac{C_V\dot{\Phi}_1^2}{2} + \frac{C_g}{2}\left(\dot{\Phi}_2 - \dot{\Phi}_1\right)^2 + \frac{C_J\dot{\Phi}_2^2}{2} + E_J\cos\left(\frac{\Phi_2}{\hbar}\right)$$

leading to classical equations of motion

$$C_J\ddot{\Phi}_2 + C_g\left(\ddot{\Phi}_2 - \ddot{\Phi}_1\right) + \frac{I_c}{m}\sin\left(\frac{\Phi_2}{\hbar}\right) = 0\,, \qquad C_V\ddot{\Phi}_1 - C_g\left(\ddot{\Phi}_2 - \ddot{\Phi}_1\right) = 0$$

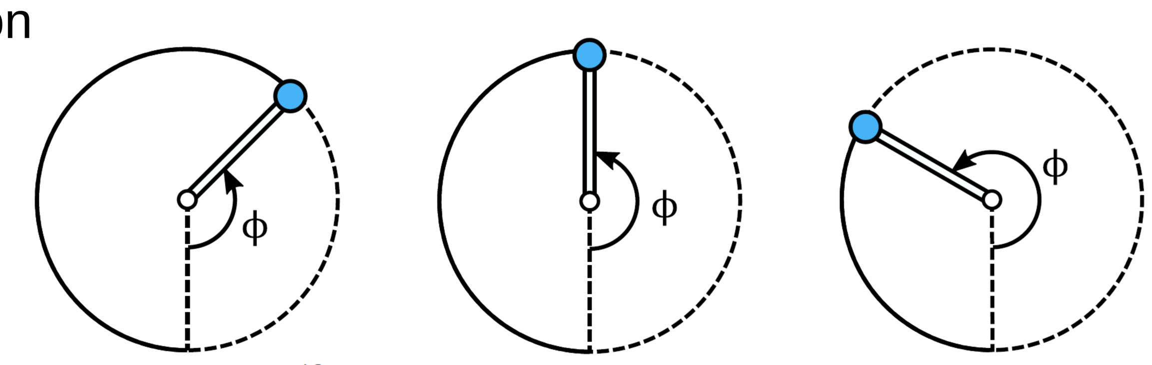
The equations of motion may be solved for the Phase across the Josephson Element, giving

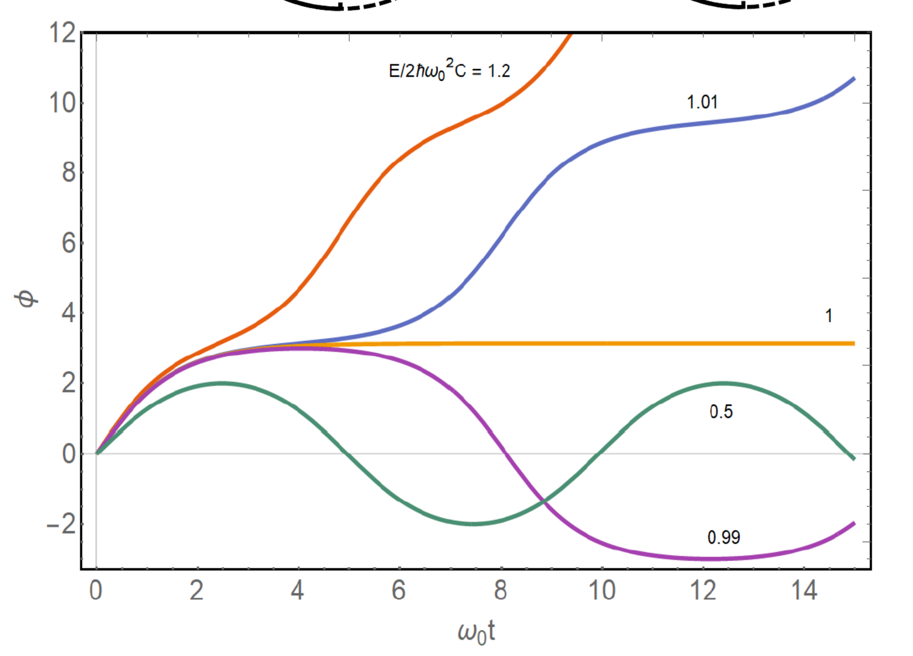
$$\ddot{\phi} + \omega_0^2\sin(\phi) = 0,$$

which is the equation of a harmonic oscillator. The small-angle approximation is invalid, but a solution is obtainable in terms of special functions

$$\phi(t) = \pm 2\,\mathrm{am}\left(\frac{\omega_0}{\kappa}\,t\ ;\ \kappa\right)$$

Where $\mathrm{am}(u\,;k)$ is the Jacobi elliptical amplitude function, and $\kappa^2 = 2\hbar\omega_0^2 C/E$, where E is the total energy of the system.





Phase ϕ across a gate-biased capacitatively-shunted Josephson junction as a function of time. The behaviour exactly models a rigid simple anharmonic pendulum, which is demonstrated by the three unique solution regimes: $\mathrm{E}/(2\hbar\omega_0^2 C) < 1$, $\mathrm{E}/(2\hbar\omega_0^2 C) = 1$, and $\mathrm{E}/(2\hbar\omega_0^2 C) > 1$, where $2\hbar\omega_0^2 C$ is the maximum potential energy storable in the junction. The behaviour mirrors that of a rigid simple anharmonic pendulum, which possesses solution regimes corresponding to the ratio of the total energy to the maximum potential energy (corresponding to the maximum height of the bob from the base).

Quantization

The Lagrangian may be used to obtain a classical Hamiltonian in terms of the numbers of helium particles in the reservoir, and the excess behind the gate diaphragm

$$\mathcal{H} = E_c\left(N - N_g\right) - E_J\cos(\phi)$$

canonical quantization is performed by promoting N and ϕ to operators with a commutation relation

$$\left[\,\hat{\phi}\,, \hat{N}\,\right] = i$$

with a Hilbert space of number eigenstates

$$\hat{N}|N\rangle = N|N\rangle$$

from the basic commutation relation, various relations may be derived. The Hamiltonian for the charge qubit results:

$$H = \sum_{N=-\infty}^{\infty}\left[E_c\left(N - N_g\right)^2|N\rangle\langle N| + \frac{E_J}{2}\left(|N+1\rangle\langle N| + |N\rangle\langle N+1|\right)\right]$$

in the regime where $E_c \approx E_J$ there is an approximate two-state system with a Hamiltonian in terms of Pauli matrices, with an offset parameter Δ.

$$H = E_C\Delta\sigma_z - \frac{E_J}{2}\sigma_x$$

Viability Estimates

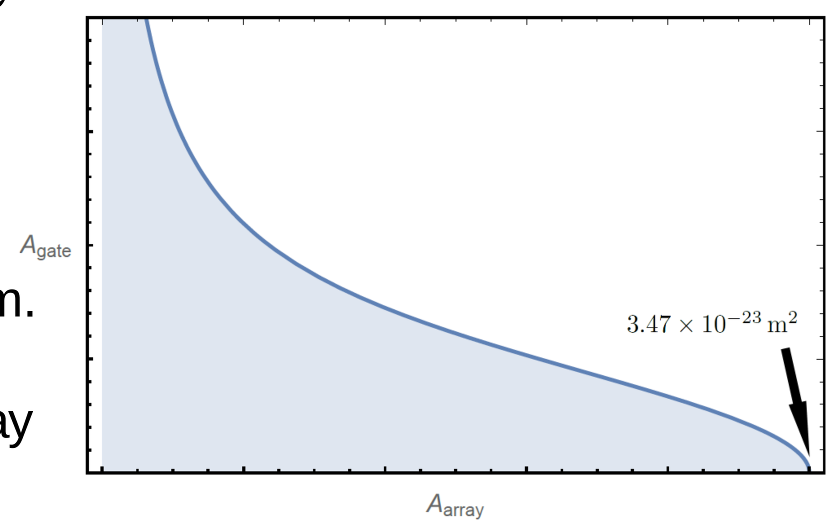
The condition $E_c \lesssim E_J$ reduces to the inequality

$$PA_{\mathrm{array}}^2 + QA_{\mathrm{gate}}^2 A_{\mathrm{array}} - 1 \lesssim 0$$

where P and Q are constants that depend on the properties of helium. This imposes a limit on the maximum size of the aperture array

$$A_{\mathrm{gate}} \lesssim \sqrt{\frac{1 - P\,A_{\mathrm{array}}^2}{Q\,A_{\mathrm{array}}}}$$

So that for a positive gate area, we must have $A_{\mathrm{array}} < 1/\sqrt{P}$.



Parameter space for 'cooper pair box' regime of qubit operation. The maximum allowable value of the array area is the x-intercept, taking a value of $3.47\times\ 10^{-23}\ \mathrm{m}^2$, which would mean an aperture width of $\sim 6\times10^{-12}\ \mathrm{m}$, or 5% of the width of a helium atom.

Conclusion

This was a project in two parts: The first was to adapt an existing framework to some different physics. This was successful, being able to derive a Hamiltonian of the expected form. The qubit design itself is flawed, however it is possible that a more complex network could achieve the two-state regime, or that a more complicated qubit type could be realized, such as the transmon-type qubit, which operates with $E_c \gg E_J$ [3].

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