Abstract

Quantum entanglement enhances the precision of the phase measurements beyond the standard quantum limit. Phase measurement using entangled states has precision bounded by the Heisenberg limit. Interferometry apparatus developed over the past century including the Mach–Zehnder and Ramsey interferometers are able to produce such precision by using non-classical states. It has been shown experimentally that by introducing entangled state into the input of these interferometer, sub-standard quantum limit precision for phase measurements can be achieved. This paper describes the experimental schemes to generate non-classical states for quantum metrology. The squeezed and the N00N states are the main non-classical states discussed in this paper. The nonlinear optical processes to generate these non-classical states are described along with the entanglement measures of these states.
1 Introduction

Metrology is the study of measurements. The goal in metrology is to search for the measurement schemes that are capable of measuring physical quantities with the highest achievable precision [1]. Due to the inconsistent in the measurement schemes using classical or semi-classical systems, it is later found that by studying the interaction of particles in a many-body system, the quantum nature of the particles could be used in the quantum metrology to generate reliable and highly sensitive measurements. Many schemes have been devised to measure the relative phase of a quantum state in a wide range of fields including trapped ions [2,3], photonics [4,5] and magnetometry in atomic ensembles (e.g. Bose-Einstein condensate [6] and diamond’s nitrogen-vacancy centres [7]). These measurement schemes are said to be able to enhance the phase sensitivity beyond the standard quantum limit.

The standard quantum limit (or shot noise limit) [8,9,10] for a precision measurement of an unknown parameter $\theta$ in a $N$-body system is defined as

$$\Delta \theta = \frac{1}{\sqrt{N}}.$$ (1.1)

This is the limit (maximum precision) that all classical separable states or mixed states can achieve. By using non-classical states, with proper configuration such that the system is not affected by decoherence, one may achieve a precision better than the standard quantum limit, or super-sensitivity [11]. The phase sensitivity for non-classical states is bounded by the Heisenberg limit [12],

$$\Delta \theta = \frac{1}{N}.$$ (1.2)

An interferometer can be used to measure the phase resolution by observing the interference pattern produced at the photon-counting detectors. Using a $N$-photon states, the detection probability $p$ that corresponds to the $N$ photon interference is periodic with respect to the relative phase $\phi$ and is proportional to $\cos(N\phi)$. Observation of such interference fringes, with period $N$ times shorter than the single photon fringe of a semi-classical state is often called phase super-resolution or phase super-sensitivity. For the remaining of the context, the term phase super-sensitivity will be used over the term super-resolution as the the term super-resolution is not exclusive to non-classical states. Phase super-resolution has been shown to be able to achieve by classical photon sources [13] while the phase super-sensitivity can only be achieved by non-classical, correlated states.

In order to quantify the estimation of the parameter e.g. $\theta$, the Cramér-Rao bound provides a lower bound on the variance of the unbiased estimator $\hat{\theta}$ such that

$$(\Delta \hat{\theta})^2 \geq \frac{1}{F(\theta)},$$ (1.3)

where $F(\theta)$ is the Fisher information [14]. The phase estimation sensitivity $\Delta \varphi$, assuming for any type of measurement is limited by the quantum Cramér-Rao bound:

$$(\Delta \varphi)^2 \geq \frac{1}{F_Q(\rho, A)},$$ (1.4)

where $F_Q(\rho, A)$ is the quantum Fisher information (QFI), $\rho$ is the input state and $A$ is a Hermitian operator associated with the measurement. For $M$ number of independent measurements, the QFI in Eq. 1.4 becomes $MF_Q(\rho, A)$. An interesting fact about QFI is that for a pure state, it is equal to four times the variance of the measurement operator $A$ associated with $\rho$, i.e. $\text{QFI} = 4(\Delta A)^2$. If the state is mixed, then QFI will always be smaller than $4(\Delta A)^2$ [15]. In other words, QFI quantifies the distinguishability of $\rho$ from the output state $\rho_\theta$. The lower bound of the phase sensitivity is estimated by obtaining the QFI associated with the state of the system. For a maximal entangled state such as the N00N state, the associated QFI scales as $N^2$, so that the quantum Cramér-Rao bound approaches the Heisenberg limit.

This paper is organised as follows. Section 2 describes the common methods used for phase interferometry with emphasis on both the Mach-Zehnder and Ramsey interferometer. Section 3 provides a brief discussion...
on the entanglement measures used to verify and quantify the degree of entanglement of the experimental generated states. Section 4 is divided into two subsections. Subsection 4.1 summaries the properties and the methods to generate and probe squeezed states. Subsection 4.2 describes the experimental schemes used to prepare entangled states like the N00N and GHZ states in both the optical and atomic regimes to achieve phase super-sensitivity. A summary of the content described in the previous sections is included in Section 5. The outlook on the advancement in phase metrology is also included in this section.

2 Interferometry

In this section, the two main types of linear interferometers used for phase metrology, namely the Mach-Zehnder and Ramsey interferometer will be addressed along with the comparisons between the two interferometers.

![Figure 1: The basic task of a linear interferometer. Phase estimation \( \theta \) is related to the evolution of a quantum state \( \rho \) under a unitary operation \( U \). The estimated phase is obtained from the output quantum state \( \rho_\theta \). \( A \): Hermitian operator associated with the measurement. Source: Ref. [16].](image)

Figure 1 describes the basic problem of a linear interferometry. In phase estimation, the output state \( \rho_\theta \) (used to estimate the phase) follows an unitary evolution of the input state \( \rho \) such that

\[
\rho_\theta = e^{-iA\theta} \rho e^{iA\theta},
\]

where \( A \) is the Hermitian operator associated with the measurement. For instance, \( A \) can be the angular momentum \( J_l \), measured by a Ramsey interferometer or the field raising and lowering operators \( \hat{a}^\dagger, \hat{a} \) which corresponds to the spatial mode \( a \) of the Mach-Zehnder interferometer. In this section, the basics of a Mach-Zehnder and Ramsey interferometry will be discussed. The common configurations of each of these interferometry techniques will be included in the discussion.

2.1 Mach-Zehnder Interferometer

The Mach-Zehnder interferometer [17, 18] is used to measure the phase difference between the two propagating light in the two arms. Figure 2 shows a basic setup for a Mach-Zehnder. Photons enter the Mach-Zehnder from the two input ports of the first beam splitter. Each output from the first beam splitter undergoes an unknown phase evolution of \( \varphi_i \) in each arm before the two beams are combined at the second beam splitter. The two outputs from the second beam splitter are detected by a pair of single photon detectors. The coincidence rate measurements from the detectors form the interference fringes are used to estimate the phase difference \( \delta \varphi = |\varphi_1 - \varphi_2| \) between the two arms of the Mach-Zehnder. Coincidence is defined as the event when both the detectors “click” or detect photon(s) at the same time.
Figure 2: Basic schematic of a simple Mach-Zehnder interferometer. Here, the input photons propagate in mode \(a\) and \(b\) are combined at the beam splitter (BS1). Each of the output of the BS1 propagates in the arm of the Mach-Zehnder before they are recombined again at the second beam splitter (BS2). Each of the output of BS2 is directed to a photon-counting detector. The coincidence count rate from the detectors by varying the relative phase shift \(\theta\) gives the interference fringes. These fringes are used to determine the phase sensitivity. \(\{a', b'\}, \{c, d\}\) are the spatial modes of the output from BS1 and BS2 respectively.

For the case where a perfectly coherent light (for example a perfect thermally stabilised laser light) is selected as the input of the Mach-Zehnder, the probability distribution \(P(n)\) follows Poissonian photon statistics \(^{19}\),

\[
P(n) = \frac{\pi^n}{n!} e^{-\pi},
\]

where the probability distribution \(P(n)\) is the probability of finding \(n\) photons from photon counting and \(\pi\) is the mean number of photons detected. The variance for the fluctuation of photon number about the mean value, \((\Delta n)^2\) is equal to the mean value \(\pi\).

All classical light with time varying beam intensity follows super-Poissonian statistics such that \(\Delta n > \pi\). This type of light source is classical and is sometimes called bunched light\(^1\). The class of light which is particularly useful in quantum metrology is the non-classical, sub-Poissonian light \(^20\). It satisfies \(\Delta n < \pi\) and photon anti-bunching\(^2\). Here, the Hanbury Brown and Twiss (HBT) setup can be used to obtain the second order correlation function \(^21\) of the incoming photons as a function of time \(\tau\),

\[
g^{(2)}(\tau) = \frac{\langle n_1(t) n_2(t+\tau) \rangle}{\langle n_1(t) \rangle \langle n_2(t+\tau) \rangle},
\]

where \(n_i(t)\) is the number of counts registered on detector \(i\) at time \(t\) and \(\langle ... \rangle\) is the expected value. \(g^{(2)}(\tau)\) is the conditional probability of detecting the second photon at time \(t = \tau\) given that a photon is detected at \(t = 0\). At \(\tau = 0\) where both photons arrive at the same time, \(g^{(2)}(0) \approx 0 < 1\) can only be obtained if the photons are anti-bunched. Thus, the coincidence measurement on HBT setup is useful to determine the state of the light whether it exhibits photon anti-bunching. In other words, it describes the tendency for the photons to arrive apart from each other. This can be clearly seen in the Hong-Ou-Mandel (HOM) dip \(^22\) \(^23\) shown in Figure 3.

The HBT setup in Figure 4 is used to obtain the correlation between the two arms consists of a 50:50 beam splitter where the emitted light is split onto two single photon detectors. The two detector outputs are connected to a time-correlated single photon counting unit and the time differences between the two signals (photo currents) are repeatedly measured to obtain the second order correlation function \(g^{(2)}(\tau)\) as a function

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\(^1\)Photons in a beam follows a time-varying intensity variation, causing photons to bunch together or coincide with each other at different intensity

\(^2\)Photon anti-bunching occurs when the photons in a beam arrives at exactly the same interval so that there are fixed spacing (in time) between photons.
Figure 3: The Hong-Ou-Mandel (HOM) dip. When there is no delay in the arrival of the photons, the two-photon interference probability $\rho_{(1,1),(1,1)}$ given by the blue dot approaches zero. This indicates photon anti-bunching. The Hong-Ou-Mandel (HOM) effect is the two-photon interference effect of a pair of correlated photons. The correlation is characterised by the second order correlation function as a function of the delay $\tau$, $g^{(2)}(\tau)$. Source: Ref. [23].

Thus by taking the photon counting results and computing the second order correlation function, the nature of the input states can be inferred [19]. The best anti-bunched statistics based on the second order correlation function is $g^{(2)}(0) = 0.11 \pm 0.18$ obtained from a squeezed coherent state by Grosse et al [24]. Photon counting has been one of the more reliable and common tools used in entanglement experiments which contributes to quantum enhancement in phase metrology. Additional to that, quadrature measurements can be done using a similar setup as the HBT setup using balanced homodyne detection as described in Figure 5. The modification includes the addition of a local oscillator field (usually derived from the same source as the signal input) into the beam splitter and the use of a subtractor module instead of the counting module. This is useful to obtain the intensity distribution of the signal beam with respect to the phase, which can be used to obtain the Wigner function, a joined spectral distribution and to reconstruct the density matrix of the state.

The most common configuration for achieving sub-standard quantum limit precision is by inserting a coherent light (any stabilised laser light) and a squeezed vacuum state into the two inputs of the Mach-Zehnder. Several experiments have successfully produced similar results by inserting a Fock state (photon number state) into each BS input but with much more experimental complexity (it is harder to create the exact Fock state required for each run). The previous configuration can be done easily with the current experimental apparatus. In particular, the squeezed state can be produced by $\chi^{(2)}$ nonlinear interaction known as frequency doubling (or second harmonic generation), in which the medium takes two photons from the pump and generates a photon with double the frequency of the pump. This is usually done in a micro resonator [25] to generate photons with high brightness. More details on the preparation of the squeezed states can be found in Sec 4.1.

2.2 Ramsey Interferometer

The Ramsey interferometer is a linear interferometer that acts on atomic ensemble. It uses microwave pulses to induce a phase change in the spin state of the atoms. It is commonly used for the atomic clock measurement, i.e. to define the unit second in terms of the hyperfine transition of a Rubidium-87 atom. It can also be used for phase metrology as the phase is related to the frequency. Figure 6 shows a common setup of a Ramsey interferometer.

Consider an atomic ensemble trapped in an optical lattice that is cooled down to few nanokelvin. As a result, Bose-Einstein Condensates are formed in the optical lattice and a coherent spin state is achieved. First, a $\pi/2$-microwave pulse is applied to the system so that the spin state of the system is an equal superposition of both the excited and the ground state. Then the state of the system is allowed to evolve under the presence of an external field for time $T$. After that, another $\pi/2$-microwave pulse is applied to the system and the projection
measurements of the z-component of the angular momentum $J_z$ are taken. If there is a frequency detuning $\delta$ between the atomic transition and the electromagnetic field, then the Bloch vector after the $\pi/2$ pulse will not be tilted exactly to the excited state, but with an angle of $\delta$. By changing the non-interaction time $T$ or the detuning $\delta$, one may observed the Ramsey fringes from the probability distribution of finding the atoms in the excited state. Figure 7 shows the expected Ramsey fringes by varying the detuning $\delta = w - w_0$ between the driving frequency $w$ and atomic resonance $w_0$.

The sensitivity of the Ramsey fringes can be enhanced by increasing the time of flight in the non-interacting zone $T$ or by reducing the average velocity of the particle inside the lattice trap. The latter can be done by cooling the particle to sub-milikelvin temperature. Here, the phase sensitivity $\Delta \theta$, in terms of the expected value of each component of the angular momentum $\langle J_l \rangle$, where $l = x, y, z$ is\[ (2.4)\]

$$\langle \Delta \theta \rangle^2 = \frac{\langle \Delta J_x \rangle^2}{\langle J_z \rangle^2},$$

where the variance of $J_x$ is defined as $(\Delta J_x)^2 = \langle J_x^2 \rangle - \langle J_x \rangle^2$.

### 2.3 Comparison between Mach-Zehnder and Ramsey Interferometers

Figure 8 illustrates the comparison between Mach-Zehnder and Ramsey interferometers. One can see that the $\pi/2$–pulse in Ramsey interferometer, which converts the atomic states to a superposition between the ground and excited states is mathematical equivalent to the action of a beam splitter in a Mach-Zehnder. Furthermore, the unitary evolution of the atomic states inside Ramsey interferometer which induces a phase change in the atomic states is similar to the relative phase induced in the Mach-Zehnder due to the difference in the optical path length in the Mach-Zehnder. Quantum enhancement in Ramsey interferometry can be obtained by preparing the atoms in a coherent, spin-squeezed state which reduces the variance of one of the components of the angular momentum similar to the squeezed light.
Figure 5: Basic schematic of a balance homodyne detection. Here, D1 and D2 are photon-counting detectors. The local oscillator, derived from the original optical source is combined with the input/probe light at the 50:50 beam splitter (and hence the term balance) to generate two output beams. These beams are detected by the detectors D1 and D2. An electronic subtractor is used to generate an output which is the difference between the photo currents from the two detectors.

### 3 Entanglement Measure

Entangled states have been used in metrology, quantum teleportation [27] and quantum cryptography [28] for the past few decades. Before one describes the methods to generate these states, it is important to address the indicators to verify if the state is entangled or at least non-classical. For squeezed states, the squeezing parameter \( r \) is usually employed to measure the degree of squeezing of the state. But how about the entangled states like the N00N, GHZ and Schrödinger cat states?

Fidelity is a way of quantifying the entanglement of a quantum state. Fidelity is defined as the statistical measure of the “closeness” of two quantum states. It is defined as

\[
F(\rho, \sigma) = \text{Tr} \left( \sqrt{\rho} \sigma \sqrt{\rho} \right),
\]

where the pure quantum state \( \sigma \) acts as a reference state to compare with the experimental constructed density matrix \( \rho \). If \( \rho = \sigma \), then associated fidelity \( F \) is 1, else the fidelity varies between 0 and 1. In order for a state to be entangled, the fidelity has to be greater than 0.5 [29, 30, 31].

The problem with the fidelity is the need of the density matrix describing the quantum state resolved at the detectors. This can be done by performing quantum state tomography [32]. In quantum state tomography, a number of different polarisation measurements are made and the results from these measurements are used to reconstruct the maximum-likelihood density matrix. From the reconstructed density matrix, one can calculate the degree of entanglement of the quantum state by using a number of tools that rely on the estimation of the quantum state. Apart from the fidelity measure, the Wigner function [33, 34] uses the reconstructed quantum state to show the entanglement of the states. The Wigner function, also known as the Wigner quasi-probability distribution is an alternative way of representing a quantum state in phase space. When the Wigner function gives negative values in the phase space, the quantum state is said to be entangled. Figure 9 shows the Wigner functions for the \( n \)-photon number state \( |n\rangle \). The Wigner function of \( |n\rangle \) has \( n \) number of zero-crossings. These functions are all radially symmetric. These states have negative Wigner function and are non-Gaussian (except for the vacuum state \( |0\rangle \) which is also a coherent state).

Bell’s inequality is a common test for entanglement. It can be done with different polarisation measurements (four is the absolute minimum) so that the results of these measurements can be fed into the Clauser-Horne-
Figure 6: A typical setup of a Ramsey interferometer. Here, the atoms are heated in an oven before they are launched into the interferometer. Initially, the atomic spin states are selected to be in the ground state $|g\rangle$. The atoms are then excited in the first cavity and their spin states are elevated into an equal superposition of both the excited $|e\rangle$ and ground $|g\rangle$ states. Then, the atoms are allowed to evolve freely in the non-interaction region between the two cavities such that the relative phase of the spin state changes with $\varphi$, which is dependent on the free evolution time. Each cavity is pumped by a microwave oscillator that provides a $\pi/2$-pulse. This causes a $\pi/2$ shift in the spin state, causing the initial spin states in the second cavity (which are in a superposition of $|e\rangle$ and $|g\rangle$) to transform into $|e\rangle$. The state detector is used to detect the phase change in the excited state of the atoms.

Shimony-Holt (CHSH) inequality [36]:

$$|S| = |E(a,b) + E(a',b) - E(a,b') + E(a',b')| \leq 2 ,$$  \hspace{1cm} (3.2)

where $a, b, a', b'$ are the different configurations of the detector settings (e.g. polarisation at four distinct angles) and $E(\ldots)$ is the expected value of the outcome of these measurements. In order to show that the state is entangled, it needs to violate the CHSH inequality, i.e. $2 < |S| < 2\sqrt{2}$.

In metrology, a parameter, for instance the phase $\varphi$ can be measured from the interference fringes from the interferometer. The interference fringes are obtained by recording the coincidence counts rate while varying the relative phase induced by the interferometer. From the interference patterns, if the visibility is greater than the maximum classical visibility $1/\sqrt{2} \approx 70.7 \%$, then the state is said to violate Bell’s inequality and hence it is entangled. The visibility $V$ is defined as

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} ,$$  \hspace{1cm} (3.3)

where $I_{\max}$ and $I_{\min}$ are the maximum and minimum intensities of the interference fringes. This method can also be used to test for eavesdropping in quantum teleportation [27] [37], as any measure on the quantum state while transferring the state from two distinct observers causes the state to undergo decoherence and the state becomes classical.

Other indicators that quantify the entanglement of a quantum state are the Schmidt number [38] and quantum entropy like the Von Neumann entropy [18]. It should be stated that either indicator has not been widely used (to date) to demonstrated experimentally the non-classical properties of the state of the system. Thus, for the following sections, indicators like the fidelity, visibility (of the interference fringes) and Wigner function will be listed instead as an evidence to the entanglement properties of the quantum state.
Figure 7: The expected Ramsey fringes (and its envelope) from the Ramsey experiment. The plot is reproduced in Mathematica based on the equations provided in Ref. [26] using the rotating wave approximation. The envelope exists due to the finite damping in the system. At each of the trough, the population of the atoms in the two levels flip. The interference fringes are used to obtain the frequency sensitivity with respect to an absolute reference, making Ramsey interferometer ideal as an atomic clock.

4 Non-Classical States

Gaussian states such as the coherent states and vacuum states have been shown to be able to produce high visibility interference fringes with sensitivity close to the standard quantum limit using the interferometers described in Section 2. For the past 30 years, many groups have devised solutions to measure the phase sensitivity beyond the standard quantum limit by using non-classical states. Among the first experimental realisation of this goal, squeezed light [39] was demonstrated in 1985. This began a global effort in implementing different schemes to improve the phase precision further.

Before extending our discussion on non-classical states, one would have to mention the coherent state, sometimes called the Glauber state. A coherent, collimated laser beam can be described by the coherent state $|\alpha\rangle$ (where $\alpha$ is an arbitrary complex number) which is defined as the displacement operator $D(\alpha)$ acting on the vacuum state, such that

$$|\alpha\rangle = D(\alpha) |0\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

where the displacement operator $D(\alpha)$ is defined in terms of the creation $\hat{a}^\dagger$ and the annihilation $\hat{a}$ operator of a harmonic oscillator, i.e. $D(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$ and $|n\rangle$ is the $n$-photon Fock state. The eigenvalue equation relating the coherent state with the annihilation operator $\hat{a}$ is $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$. The uncertainties of the coherent state are equal for position $x$ and momentum $p$ quadratures and they saturate the Heisenberg uncertainty relation such that $(\Delta x)(\Delta p) = 1$. The average photon number is the absolute square of the amplitude, i.e. $\langle n \rangle = |\alpha|^2$. The probability distribution for finding $n$-photon state follows the Poissonian statistics with the variance of obtaining the photon number $n$ equals to the average photon number, i.e. $(\Delta n)^2 = \langle n \rangle = |\alpha|^2$.

Despite the fact that coherent state is one of the easiest states to generate experimentally, the uncertainty scales as $1/\sqrt{n}$ such that its measurement precision is limited by the standard quantum limit. Thus, it will be necessary to employ different schemes to enable quantum enhanced measurement. Here, this section will focus on two of the more common non-classical states studied in the literature: squeezed states and N00N states. Alternative quantum states such as the GHZ and Holland-Burnett states [40] are compared to these states. These states have to fulfil the following conditions before they can be viable in metrology: robust to losses.

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4The coherent state is named after the Roy J. Glauber who is one the recipients of the Nobel Price in Physics in 2005 for his contribution on quantum coherence.
high purity, long storage time for complete measurements and with phase sensitivity higher than the current metrology standard.

4.1 Squeezed state

4.1.1 Definition

Phase measurements using squeezed state, or more precisely the squeezed coherent state have been shown to be able to beat the standard quantum limit while saturating the Heisenberg uncertainty. The degree of squeezing is usually expressed in terms of the reduction of noise in a certain quadrature. The quadrature can be either the amplitude, which contributes to the amplitude-squeezed light, the momentum, i.e. phase-squeezed or both, i.e. quadrature-squeezed. The phase-squeezed state is used to reduce the variance in phase which is particularly useful for phase metrology, and hence contributes to the precision improvement in the detection of the gravitational waves [41] and the atomic clock [42]. It can be done on different platforms apart from the optical regime, e.g. in optical lattices, trapped ions/atoms in cavities and in fibre optics. These methods will be described in Section 4.1.3.

Using the same convention as the coherent state, a single mode squeezed state \( |\alpha, r\rangle \) acting on a vacuum state \( |0\rangle \) is

\[
|\alpha, r\rangle = \hat{D}(\alpha)\hat{S}(r) |0\rangle ,
\]

where the squeezing operator \( \hat{S}(r) \) is defined in terms of the complex squeezing parameter \( r \) (where \( r = |r|e^{i\theta} \) and \( \theta \) is the squeezing angle) such that \( \hat{S}(r) = \exp \left( \frac{1}{2}(r^*\hat{a}^2 - r\hat{a}^2) \right) \). The special case for this kind of squeezed states is the squeezed vacuum state that posses zero mean values in both the quadratures, i.e. \( \langle \hat{X}_i \rangle = 0 \). This class of squeezed state is more commonly used in metrology than the squeezed coherent state due to the simplicity to produce such states experimentally. For \( \alpha = 0 \), the squeezed vacuum state can be defined as

\[
|r\rangle = \hat{S}(r) |0\rangle .
\]

The squeezed vacuum state \( |r\rangle \) can also be written as a superposition of the even photon number states:

\[
|r\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0, \text{even}}^{\infty} \frac{H_n(0)}{\sqrt{n!}} \left( \frac{\tanh r}{2} \right)^{n/2} e^{in\theta/2} |n\rangle ,
\]

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(a) Wigner function of a vacuum state $|n = 0\rangle$ in phase space. Notice that it has a Gaussian waveform and is non-negative anywhere.

(b) Wigner function of a three-photon-state $|n = 3\rangle$ with a negative trough at the centre in phase space.

(c) Wigner function of a four-photon-state $|n = 4\rangle$ with a positive peak at the centre in phase space.

Figure 9: Wigner functions of $n$-photon Fock states with $n = 0, 3, 4$. These plots are generated using the QuTiP package in Python [45]. For an even number Fock state, there is a peak at the centre while a negative trough is located at the centre for an odd number Fock state. Note than in general, the Fock states is non-Gaussian, i.e. they can take negative values, with the exception being the vacuum state.
due to the fact that the odd \( n \) Hermite polynomial is antisymmetric and has \( H_{\text{odd},n}(0) = 0 \). The photon counting statistics of the squeezed vacuum state follows the Boltzmann distribution with the mean number of photons, \( \langle \hat{n} \rangle = \sinh^2 r \), which is non-zero despite what its name suggests.\(^5\)

Figure 10 shows the Wigner functions for coherent state, squeezed coherent state and squeezed vacuum states. Note that neither of these states has a negative Wigner function. However, the squeezed states (vacuum and coherent), both having uncertainty in certain quadrature smaller than that of the vacuum state, are non-classical despite having a positive Wigner function. This is due to the fact it has been proven in Ref. \(^{13}\) that the Wigner function of a pure quantum state is non-negative if and only if the state is Gaussian. These states are Gaussian and thus their Wigner functions are non-negative. Thus, one can formulate the nonclassicality of a quantum state: a state is non-classical if its Wigner function is negative or if the variance of any quadrature operator is smaller than its value in the vacuum state.

The two-mode squeezed vacuum state or a twin-beam state is the simplest two mode non-classical Gaussian state which is defined as

\[
|\text{TMSV}\rangle = S_2(r) |0,0\rangle ,
\]

where the two-mode squeezed operator is \( S_2(r) = \exp \left( r^* \hat{a} \hat{b} - r \hat{a}^\dagger \hat{b}^\dagger \right) \). \( \hat{a} \) and \( \hat{b} \) refer to operators associated with mode \( a \) and \( b \). It can also be written as \(^{44}\)

\[
|\text{TMSV}\rangle = \sum_{n=0}^{\infty} \frac{\tanh^n r}{\cosh r} |n,n\rangle
\]

in the Fock basis. A notable feature of this class of states is that despite the difference in modes \( a \) and \( b \), the state is a superposition of the terms with the same photon number in each mode. In the limit where \( |r| \to \infty \), the state will transform into the EPR state, i.e. \( \frac{1}{\sqrt{2}} \left( |0,0\rangle_{a,b} + |1,1\rangle_{a,b} \right) \). This has been realised experimentally \(^{15}\) by interfering two input squeezed states at a 50:50 beam splitter with a relative phase of \( \pi/2 \). From the same reference, they have shown that interference fringes of two-mode squeezed vacuum is sensitive to the intercavity relative phase while the EPR entangled state depends on the initial phase of the input states. The two-mode squeezed vacuum is also comparable to the superposition of the coherent states with opposite phase which is essentially a Schrödinger cat state \(^{46}\).

4.1.2 Preparation

Squeezed states have been created since 1985, first using nondegenerate four wave mixing \(^{39}\) and a year later, using spontaneous parametric down-conversion to achieve squeezing of more than 50% compared to the vacuum state \(^{47}\). As described in Section 2.1, the squeezed state can be generated using Second Harmonic Generation (SHG), showing a trade-off between bright amplitude and relatively low degree of squeezing compared to other optical parametric generation processes. Squeezed light can also be generated inside fibre optics by utilising the Kerr nonlinearity and four-wave mixing inside a fibre loop, in semiconductors and by atom-photon interaction which contributes to spin squeezed state.

One of the most common ways of generating a squeezed state is by nonlinear optical wave mixing processes, in which pairs of photons are emitted either into degenerate (same) or nondegenerate (different) modes. The nondegenerate case denotes the production of two-mode squeezing states proposed in Ref. \(^{46}\), which resembles the Schrödinger cat-like state, but easier to generate experimentally compared to the cat state. Here, the nonlinear optical processes for generating these squeezed states are discussed in more detail in this section, with emphasis on spontaneous parametric down-conversion and four-wave mixing.

Spontaneous parametric down-conversion (SPDC) is a second order \( \chi^{(2)} \) nonlinear optical process in which a pump photon injected into a nonlinear medium (e.g. beta-barium borate crystal) produces two photons with

\(^5\)Here, vacuum does not mean that there is no photons in the state (on the contrary with the vacuum state mentioned in Section 2.1 where there is literally no photon input into the Mach-Zehnder), it merely states that the state has an uncertainty in a certain quadrature which smaller than that of a vacuum state.

\(^6\)EPR (Einstein-Podolsky-Rosen) paradox suggests that a wave function does not denote a complete description of a quantum state. This gives rise to the hidden-variable theory which is later known as the Bell’s theorem.
(a) Wigner function of a coherent state with amplitude of $\alpha = 2 + 2j$ in phase space. Notice that the uncertainty in each quadrature is the same and thus it forms a uncertainty circle that satisfies the Heisenberg uncertainty relation.

(b) Wigner function of a squeeze vacuum state in phase space, with squeezing parameter $r = 0.5$. Notice that the uncertainty in one quadrature is less than the other. However, their uncertainties still fulfil the Heisenberg uncertainty relation.

(c) Wigner function of a squeeze coherent state in phase space, with squeezing parameter $r = 0.5$. In comparison with the squeezed vacuum state, the squeezed coherent state is merely the squeezed vacuum state acted by a displacement operator such that it is displaced in the phase space by an amplitude of $\alpha = 2 + 2j$.

Figure 10: Wigner functions of coherent state, squeezed vacuum state and squeezed coherent state. Among these three, the squeezed states (vacuum and coherent) are non-classical due to the fact that one of their quadratures has uncertainty lower than that of the vacuum state.
Figure 11: Schematic of parametric down-conversion using a $\chi^2$ nonlinear crystal. The top figure shows the down-conversion process where a photon pumped into a nonlinear medium gives out a photon pair (labeled signal and idler). The polarisation of the output photons are represented by the two rings. For the i) Type I phase matching, the photon pair has the same polarisation, i.e. they are both either vertical $|V\rangle$ or horizontal $|H\rangle$ linear polarised. For the ii) type II phase matching, both photons have opposite polarisations, one in $|H\rangle$ and another in $|V\rangle$. The intersection between the two circles is unpolarised and it displays polarisation entanglement after proper phase matching (to compensate for the phase induced due to birefringence).

frequencies $w_1, w_2$ which satisfy energy conservation such that $w_p = w_1 + w_2$, where $w_p$ is the pump frequency. For the degenerate case, the two output photons have the same frequency, i.e. half of the pump frequency and otherwise for the non-degenerate case. The SPDC process can be further classified into 2 cases: Type I phase matching, where both photons have the same polarisation and Type II phase matching where both photons have opposite polarisation. A single mode squeezed state is produced as a consequence to the Type I SPDC. The two-mode squeezed state is created via Type II SPDC in which the photon pair is mode-entangled. Figure 11 shows the graphical representation of the light rings describing each phase matching process.

Four wave mixing is a $\chi^{(3)}$ nonlinear process in which two or more intense pump photons interact inside a nonlinear medium to produce two entangled photons. This is described in Figure 12. It can be done inside a Kerr medium (e.g. inside a fibre). It bears an advantage over the SPDC method as it can be done on a relatively smaller scale that the SPDC case. It has been done inside a cavity embedded on a chip to produce high purity entangled photons. Creating photon pairs (bi-photons) on a chip-scale micro resonator reduces the cost and power to produce and hence offers scalability for multipartite systems. Using an on-chip silicon ring resonator, path entangled bi-photons (which is also a two-mode squeezed state) are generated with a visibility of $93.3 \pm 0.2\%$ [49]. Numerical and analytical analysis of the dispersion profile in an azimuthal optical micro resonator as well as the spectra profile of the bi-photons produced due to the four wave mixing in the on-chip resonator are described in Ref. [50].

Any of these two processes leads to a $\chi^{(2)}$ or $\chi^{(3)}$ (for Kerr mediums) nonlinear process known as optical parametric amplifier (OPA). Figure 12 shows the possibility of achieving optical parametric gain using four-wave mixing which leads to the construction of an OPA. Due to the reliability of this method to produce high purity and visibility bi-photons plus the ability to control the squeezing parameters and create large squeezing, it has became one of the most common methods to create squeezed state. This method can be used to generate high optical power signal beam without applying too much power on the pump, which is usually not possible without amplification due to the low down-conversion rate of the nonlinear crystal $\sim 10^{-10}$. Instead of just a pump as the input of a nonlinear medium, an input seed is required to reduce the required intensity for the same gain (for the case where there is no seed, a higher input intensity is require to obtain the same gain) to produces bi-photons (signal and idler). It can also be done in a process called the optical parametric oscillation where
it uses a cavity to resonate the generated waves and the latter acts as its own input seed. OPA is used to overcome the fact that the output from the SPDC has a relatively weak intensity and spread out phase/wave vector. Figure 13 shows the schematic of an OPA involving a seed and a pump source.

In order to achieve high $N$, it is common to amplify the squeezing process of atomic ensemble using cavities. Slusher et al. [39] managed to obtain 0.3 dB squeezing below the standard noise limit by trapping sodium atom vapour in two Fabry-Perot cavities. They employed a four-wave mixing scheme which exploits the Raman oscillation between ground and excited states of the sodium atoms using an optical pump. This paper marks the stepping stone for other groups to improve on this method to produce larger squeezing. Although it is much more convenient to place an atomic ensemble between the two mirrors of the Fabry-Perot cavity, its strict requirement for having highly parallel mirrors and its low $Q$ (quality factor) make it less than ideal to perform long term measurements on the generated squeezed states. The alternative to this is the whispering-gallery-mode resonators described in Ref. [25, 51, 50] which have high-$Q$ ($Q \geq 10^8$), high finesse and large circulating power. Despite the difficulty of trapping large number of atoms due to the geometry of the cavity, it has more potential in metrology than the Fabry-Perot cavity due to the higher squeezing and longer photon lifetime.

Squeezing occurs due to the optical nonlinearity in the medium. The process where an atom interacting with an optical wave resonant with one of its transitions is nonlinear. Figure 14 shows the mechanism that leads to the generation of squeezing. Consider a $\Lambda$-shaped atomic energy level configuration with two ground states with energy difference of $\hbar \Delta$ coupled to an excited state by optical transitions. Suppose the atom is in one of the ground states. A pump with frequency $w$ excites the ground state to the excited state $|a\rangle$ through Raman scattering which results in the emission of Stokes photon with frequency $w - \Delta$. The other ground state is also excited by the pump photons which results in the emission of an anti-Stokes photons with a frequency of $w + \Delta$. The two emitted photons will then undergo four wave mixing and produces a two-mode squeezed states which comprises of the Stokes and anti-Stokes photons.

The main issues with this type of squeezing is the incoherent emission due to the nature of the spontaneous four-wave mixing, i.e. the possibility of introducing radiations that do not belong to Raman scattering and dephasing due to the thermalisation of the optical state. These result in the lower squeezing compared to their photonics counterpart.
Figure 13: Optical parametric amplification of the signal photons using a pump and seed source. By adding an extra seed input, from the energy conservation, where \( w_{\text{pump}} + w_{\text{seed}} = w_{\text{residual}} + w_{\text{signal}} + w_{\text{idler}} \), the wavelength of signal photons increases compared to the optical parametric generation case (no seed input), hence reduces the required pump intensity as in the optical parametric generation case for the same gain. Amplification occurs when some of the pump photons are converted into the signal and idler photon pairs which causes an increase in \( w_{\text{signal}} \). Any phase mismatch will degrade the performance of the amplification. Phase matching, i.e. conservation of the momentum has to be satisfied or at least maximized to enable optical amplification.

4.1.3 In Atomic Ensemble and Fibres

The squeezing of the atomic spin states for Bose-Einstein condensate in optical lattices has been demonstrated experimentally in Ref. [52, 53, 54]. Among these, Christian Gross [52] and his group are the first to use spin squeezed atoms to improve the precision of the interferometry. First, a few hundred rubidium atoms are trapped in a 1D optical lattice. They then laser-cool the atoms to the temperature of the order of 10 nK to form a Bose-Einstein condensate (BEC), where all of the atoms are in the same quantum state. Then a magnetic field is carefully applied to the BEC so that there is a Feshbach resonance that facilitates the interaction between atoms, making a transition between the two hyperfine levels. It acts as a nonlinear beam splitter such that the output is a coherent spin squeezed state, a mode-entangled quantum state. Figure 15 shows the nonlinear interaction in the atomic ensemble which contributes to the generation of spin-squeezed state. First, a fast \( \pi/2 \) pulse applied to the BEC produces a coherent spin state with the average angular momentum in the \( z \)-direction \( \langle J_z \rangle = 0 \). Then the coherent spin states are allowed to undergo evolution under the interactions, causing a sheering effect which reduces the relative phase fluctuations, so that the circular uncertainty region transforms into elliptical. As a result, a coherent spin squeezed state is created with the squeezing direction forming an angle \( \theta \) relative to the \( J_z \). Finally, a \( \pi/2 \) pulse is applied to the BEC to rotate the uncertainty ellipse around the centre by \( \pi/2 \) to prepare a phase squeezed state. Observing the Ramsey fringe, they managed to obtain 15\% improvement in phase sensitivity and a squeezing factor of 8.2 dB below the classical measurements. They managed to create entanglement of 170 atoms. Muessel and his group [6] succeeded to improve on this scheme by generating a much larger many-body entangled system of 12,300 atoms but with a lower degree of squeezing, i.e. 5.3 ± 0.5 dB below the classical fluctuations.

The experimental scheme to prepare spin squeezed state by trapping atoms inside cavities is described in Ref. [50]. The modified Ramsey interferometer used in their experiments described in Figure 16 consists of two microwaves cavities which are cooled down to 0.8 K to reduce thermal fluctuations. The Rubidium atoms are prepared in the circular Rydberg state \( |g⟩ \) with principal quantum number \( n = 50 \). The cavity is detuned from the transition between \( |g⟩ \) and \( |e⟩ \) (circular state with \( n = 51 \)). The pulsed atoms are sent through the cavity \( C \) which is sandwiched between the two microwave cavities with both applying a resonant \( \pi/2 \)-pulse to the
Figure 14: Four wave mixing in an atomic Λ-system which leads to the emission of Stokes and anti-Stokes photons (similar to signal and idler photons in parametric generation). Two mode squeezing occurs if both photons are coherent, i.e. have the same frequency and phase. Source: Ref. [44].

atoms. The cavity $C$ is irradiated by a coherent microwave pulse which creates a field of controlled amplitude and phase inside the cavity. This scheme, modified by Hosten et al has reported a spin-squeezed with noise level about $20.1 \pm 0.3$ dB lower than the standard quantum noise level for half a million of rubidium atoms via an optical-cavity based measurements.

Fibre is made up of silica, a highly nonlinear, Kerr medium. The refractive index of a Kerr medium (fibre) varies with the square of the electric field component or linearly with the intensity $I$ of the propagating light such that $n = n_0 + n_2 I$, where $n_0$ is the zero intensity reactive index of the medium and the coefficient $n_2$ is related to the third order nonlinear susceptibility $\chi^{(3)}$. Thus, the phase of the propagating light inside the fibre varies with its intensity, resulting in squeezing. Figure 17 shows the cross-section of a polarisation-maintaining fibre which serves to preserve the polarisation of the light in both the fast and slow axes.

Exploiting the birefringence of the fibre, where beams with different polarisation relative to the fast or slow axis of the medium will experience different refractive index and propagate in different speed, Sagnac type interferometer is used to measure the relative phase of two beams propagating inside the fibre loop. The basic schematic for the interferometer in Figure 18 is first described in Ref. [57] where a laser pulse impinges at a beam splitter splits into two beams: a strong pulse and a weak pulse which are then coupled to a PM fibre, launched into the two ends of a fibre loop. Due to the Kerr nonlinearity, the strong pulse acquires an intensity dependent phase shift and the weak pulse is unaffected by the nonlinear effect. This causes the initial circular uncertainty area to transform into an ellipse. The two pulses propagate inside the fibre before they merge at a beam splitter. By choosing the input polarisation about $45^\circ$ with respect to the optical axes of the fibre, the two independently squeezed beams in $s$ and $p$ polarisation can be generated simultaneously, creating a phase squeezed state at the output of the polarising beam splitter (PBS). Balance homodyne measurements can be used to detect the relative phase of the beams by including a local oscillator and photon-counting modules (where their electronic signals are subtracted). The resulting photo current from the detectors exhibits the phase sensitive squeezed noise.

Apart from all these sources, there are still many ways where new approaches are invented to cope with the weakness of each methods. One of them is the nitrogen-vacancy (NV) centres in diamonds. This method is widely used as a single photon sources but it has been shown to be able to act as a hybrid optomechanical systems, containing both a NV centre and a mechanical oscillator [58]. NV centres in diamond are usually
known as artificial atoms in solid systems which consists of a nitrogen atom and a vacancy at the nearest neighbouring site (negatively charged and has spin 1 in the ground state). They portrays advantages in quantum information, which includes the ability to prepare and read-out the spin state at room temperature as their electronic spins have a relatively long coherence time of few ms at room temperature (due to the weak spin-orbit coupling of diamond and low concentration in carbon-13). Spin-phonon interactions in the NV ensemble simulate spin squeezing. It is useful for magnetometry \cite{7}, as a highly sensitive magnetic field sensor with resolution nT/√Hz \cite{59} \cite{60}. This enables magnetic field sensing below the projection shot noise level \cite{61}. Spin-photon (polarisation) entanglement in the NV centres in diamonds has been shown in Ref. \cite{62} to exhibit a sufficient degree of entanglement with a lower bound of 0.69 ± 0.07 in the fidelity, which is greater than its classical value, 0.5.

### 4.2 Entangled state

The two-mode squeezed vacuum (TMSV) mentioned in Section 4.1 shows continuous variable (CV) entanglement (entanglement in continuum of states like position and momentum) in a photon pair generated from spontaneous parametric down-conversion. It has been shown to be able to completely transfer a quantum state (it can be the photon number or polarisation) of an optical mode across physical distances which contributes to the teleportation of qubits \cite{27}. The generation of entanglement photonics EPR states from the two-mode squeezed vacuum on-chip has been reported to be able to achieve phase sensitivity 1.44 ± 0.12 dB below the shot noise level \cite{63}.  

Figure 15: Basic schematic of a nonlinear interferometer used in Ref. \cite{52}. Using a nonlinear beam splitter, coherent spin squeezed states are created. These states undergo a relative phase change $\varphi$ in the interferometer. Since the coherent spin squeezed states have reduced uncertainty in relative phase, phase super-sensitivity can be achieved beyond the standard quantum limit. Here, a Bloch vector on the Bloch sphere, representing the state of the system is displayed for different phases in the interferometer. By measuring the $z$-component of the angular momentum, the phase sensitivity can be obtained using Eq. 2.4.
Figure 16: An example of the Ramsey interferometer from Ref. [55]. An atomic Ramsey interferometer (auxillary cavities $R_1$ and $R_2$) sandwitches a superconducting Fabry-Pérot cavity C. The pulsed source $S'$ induces $\pi/2$-pulses resonant to the transition between the ground and excited states of rubidium-87 atoms in $R_1$ and $R_2$ with relative phase $\phi$. The atoms are excited to circular Rydberg state in $B$ with principal quantum number 50 and with velocity $250 \pm 1$ m/s. The actuator $S$ feeds the cavity $C$ to set up the quantum feedback. Here, K: feedback controller, D: field-ionisation detector, A: microwave switch, $\Phi$: phase shifter, $S$: actuator source.

One of the modifications to the EPR entangled state is the introduction to the N00N state. A N00N state with $N$ maximal path-entangled photons which acquires a phase $\phi$ with $N$-fold phase super-resolution is defined as

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|N, 0\rangle + e^{-iN\phi} |0, N\rangle),$$  

(4.7)

where $|N, 0\rangle = |N\rangle_a |0\rangle_b$ denotes a $N$-photon Fock state in mode $a$ and vacuum state in the other. The N00N states, up to $N = 4$ [4, 64] and $N = 5$ [64] number of photons have been demonstrated but it has been found to be intolerant to losses, i.e. a single photon loss in the lossy medium will destroy the phase super-sensitivity. The phase uncertainty derived from a perfect, loss-free N00N state scales as the Heisenberg limit, $1/N$. Figure 19 shows the comparison between a single fringe and $N = 5$ interference fringes. The probability of detecting $N$-photons derived from the multiphoton interference fringes which varies with phase $\phi$, $P(N, \phi)$ is

$$P(N, \phi) = (1 + \cos N\phi)/2.$$  

(4.8)

The same sensitivity can be achieved by the Greenberger–Horne–Zeilinger (GHZ) state [65]. This state is analogous to the N00N state but it is particle entangled instead of the path entangled as in the N00N state. The GHZ state is defined as

$$|\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes N + |1\rangle \otimes N),$$  

(4.9)

which is a superposition of the $N$-tensor product of $|0\rangle$ and $|1\rangle$. These states can be the eigenstates of $z$-component of the angular momentum, $J_z$ in the atomic ensemble, i.e. with $|0\rangle$ and $|1\rangle$ correspond to the spin-$(1/2)$ and spin-$(1/2)$ states respectively. For $N = 3$, the corresponding GHZ states that evolves under unitary dynamics is [66]

$$|\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}} (|000\rangle + e^{-i\varphi} |111\rangle),$$  

(4.10)

where $\varphi$ is the relative phase of the two terms. Similar to the N00N state, the phase sensitivity approaches the Heisenberg limit.

The first N00N state is a two-photon N00N state which is demonstrated experimentally in 1990 by Rarity et al. [17] and Ou et al. [68] from optical parametric down-conversion. It is generally easier to generate even

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It is called a N00N state as that the wavefunction is a superposition of the two-mode Fock state with $N$ photon number in either modes.
Figure 17: Cross-section of a panda-type polarisation-maintaining fibre (PM fibre). The stress rod at the opposite sites of the core inside the cladding are there to induce birefringence in the fibre. Birefringence occurs inside the fibre such that photons with opposite polarisation travel with different velocity inside the fibre due to the differences in the refractive index for the two polarisations. The photons inside a PM fibre travel either in the fast or slow axis, where the refractive index for the fast axis are lower than the slow axis and hence photons move faster in the fast axis (and hence the term fast). As a result, the birefringence of the fibre locks the two orthogonal modes of the light into two birefringence axes to preserve their polarisation.

$N$-photon $N00N$ state using the photon pairs generated from parametric down-conversion sources. However, Mitchell et al have shown that it is not impossible to generate odd number $N00N$ states \[^{67}\]. Their method is illustrated in Figure 20. They generate a photon pair from a pulsed Type II-parametric down-conversion pumped by a Ti:Sapphire oscillator (average optical power: 50 mW, central wavelength: 405 nm) so that the photon pairs are orthogonally polarised. The local oscillator originates from the attenuated pump laser. These three photons (one from the local oscillator and two from parametric down-conversion) transform into the state \( |3,0\rangle_{H,V} \) at the coincidence detector. When the two photon pairs merge at the polarising beam splitter, it produces a DC photon in a single mode. The polarisation of the output from the polarising beam splitter is then rotated by $\pm 45^\circ$ by half-wave plate, followed by a $\pm 60^\circ$ rotation by the partial polariser (PP) given that there are no photons being reflected into the dark ports. The DC photons and the local oscillator photon are then combined at the last interface of the PP, putting all three photons into the same spatial mode given that there are no photons exiting at the dark ports. These photons are passed through the quarter-wave plate to transform their state into the three-photon $N00N$ state and then a phase shifter to induce a relative phase shift in the final state. There, three $\pm 45^\circ$ linear polarisers are used to detect the phase shifted behavior of the entangled photon. The visibility\[^8\] of the interference fringes gives 42 $\pm$ 3 %. They have also shown that their method of post-selection and combining photon pairs with the local oscillator(s) can be adapted to generate multipartite entanglement.

A "heralded scheme" \[^{70}\] can be used to create $N00N$ state up to $N = 4$. An example of the heralded scheme used to generate $N00N$ state is described in Figure 21. Brian Smith and his group reported in Ref. \[^{69}\] the generation of a high purity two-photon $N00N$ with 63% lower-bound visibility from heralded single photon sources. To created the heralded two-photon $N00N$ state, two spontaneous parametric down-conversion sources are simultaneously pumped to create two photon pairs. One photon from each photon pair is directed to two single photon detectors. If both of the detectors click, the detection of the trigger photons heralds the generation of the sibling photons, which are then combined at a 50:50 beam splitter resulting in the a heralded two-photon $N00N$ state. The combination of a phase shifter in one of the arms of the Mach-Zehnder and a second beam

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\[^8\]The visibility is related to the purity of the photons and the mode overlap of the states.
Figure 18: Basic Schematic of a Sagnac interferometer. The input photons (LP) are split at the first beam splitter such that 30% of the photons are separated and are treated as a local oscillator (LO) or a reference beam in the homodyne measurement. The remaining photons are then directed to a balanced beam splitter (BS1) where two beams are split and coupled into the fibre loop. The two beams are then superimposed inside the fibre loop before a single output (SV) is sent to the second beam splitter which is then recombined with the LO to perform homodyne measurement. The piezzo mirror (M) is used to alter the length of the arm of the interferometer. The relative phase is dependent on $\Delta \varphi = \frac{2\pi LD}{\lambda v}$, where $L$: length of the fibre coil, $D$: diameter of the coil, $\lambda$ of the LO/input and $v$: speed of the light traveling in the fibre. The interference fringes obtained from the difference in photo currents (homodyne measurements) give the sensitivity of the phase measurement. Source: Ref. [57].

Other methods have been used to generate N00N states. One of them [23] employs the reverse Hong-Ou-Mandel (HOM) effect\(^9\) to generate high fidelity entangled two-photon N00N states which are loss tolerant. They used a heralding scheme similar to that in Figure 21 to prepare the required quantum state and a homodyne detection scheme to perform state tomography. An on-chip silicon ring resonator partnered with an integrated Mach-Zehnder has been shown to be able to produce entangled two-photon N00N state [10]. Low noise bi-photons are produced in the ring resonator via spontaneous four wave mixing induced by tuning the pump’s frequency to the resonance of the ring resonator which are then allowed to interfere in the on-chip beam splitter. They managed to obtain a high fidelity of at least 88% after post-selection. A second order N00N state has been reported to be generated from spin-wave interferometry in Ref. [71]. However no phase super-sensitivity is observed due to the proximity of the coherence time of the spin wave to evolution time.

There is a trade-off between the scalability and visibility of the interference fringes. The visibility of the interference fringes for multi-photon coincidence rate decreases sharply at high photon number, particularly when three or more photons are involved. However, by clever and careful manipulation of setups and the sources for generating photon pairs, high purity, brightness and visibility can be achieved simultaneously as described by Afek et al [64]. This experimental scheme, which mixes both quantum state of light and coherent state to generate N00N state is described in Figure 22.

Afek et al managed to generate up to $N = 5$-photon N00N state (42 ± 2% visibility) with non-classical fidelity ($> 0.5$) in the multiphoton interference. A Ti:Sapphire pulse laser with 120 fs pulse is pumped into a

\(^9\)The HOM effect is the quantum interference phenomena where two indistinguishable photons that are overlapped at a symmetric beam splitter always emerge in the same output mode.
beam splitter which divides the pulse into two paths. One of them is attenuated to generate a coherent state with vertical linear polarisation $V$. The other pulse is pumped into a lithium triorate (LBO) crystal to undergo frequency doubling to generate 404 nm pulse. This pulse then undergoes Type I spontaneous parametric down-conversion (SPDC) to generate SPDC photon pairs. The quantum SPDC light (polarisation $H$) then recombines with the coherent light (polarisation $V$) at a polarisation beam splitter to generate N00N state. The output is then passed to the Mach-Zehnder interferometer to be detected by $N$-number photon-counting detectors. Up to five photons coincidence rates are measured to obtain the interference fringes. The visibility has a lower bound of 73% for the $N \leq 4$ photons N00N states. This method has several additional advantages over the heralded scheme in the way that no post selection or state projection are required, high brightness SPDC sources are not required due to the addition of the coherent state and it works for arbitrary $N$-entangled photons without much alterations to the setup (except the usage of detectors to resolve $N$-photons interference fringes). High fidelity N00N state can be obtained by optimising the pair production ratio parameter, $\gamma = |\alpha|^2/r$ of the setup by varying the amplitude of the coherent state $|\alpha|$ and the squeezing parameter $r$.

GHZ states has been observed in 1999\[66\] using polarisation-entangled photon pairs generated from SPDC. Three photon GHZ state is generated by transforming two pairs of photon pairs into three entangled photons and one trigger photon. The setup used in the literature is described in Figure 23. The Ti:Sapphire laser generates high energy pulses with period $\approx 200$ fs at wavelength $\lambda = 394$ nm. These pulses are pumped into a nonlinear crystal (beta-barium borate) to generate photon pairs in the following form ($H, V$ refer to the horizontal and vertical polarisations respectively in mode $a$ or $b$): $\frac{1}{\sqrt{2}}(|H\rangle_a |V\rangle_b - |V\rangle_a |H\rangle_b)$. The photon pair in arm $a$ propagates towards a polarising beam splitter that transmits the $H$-polarised photons towards the detector $D$ and reflects the $V$-polarised photons to the trigger $T$. In arms $b$, the photon pairs are directed to a 50:50 beam splitter (polarisation independent). One of the outputs from each beam splitter is sent to the final polarising beam splitter. A half-wave ($\lambda/2$) plate is placed in between the two polarising beam splitters so that the $V$-polarisation is rotated by $\pi/4$. The rest of the outputs are sent to a interference filters before being directed to the single photon detectors $D_1$, $D_2$ and $D_3$. When a four-fold coincidence is detected, the state detected by the coincidence recordings of $D_1$, $D_2$ and $D_3$ given that the polarisation of the photon at the trigger $T$ is horizontal, is a superposition of the two detection outputs (i.e. a GHZ state):

$$\frac{1}{\sqrt{2}}(|H\rangle_1 |H\rangle_2 |V\rangle_3 + |V\rangle_1 |V\rangle_2 |H\rangle_3).$$ (4.11)
Figure 20: Schematic in generating three-photon N00N state. A photon pair generated by type II spontaneous parametric down-conversion is sent to a polarisation beam splitter (PBS) to generate a polarisation-entangled state at the output of the PBS. The polarisation-entangled state is rotated by $\pi/4$ using a half-wave plate (HWP), followed by the partial polariser (PP). The DC photons are then combined with a local oscillator (LO) photon at the last interface of PP, making three photons to be in the same spatial mode as long as there is no loss at the dark ports. A quarter-wave plate (QWP) is then used to transform the state into a three-photon N00N state. A phase shifter is then used to induce a relative phase $\phi$ in the N00N state which is used for phase sensitivity measurements. Here, post-selections of the desired states (select states that do not leak to the dark ports) occur in the dark ports and partial polariser (PP). Source: Ref. [67].

The observed visibility from the four-fold coincidence rate-delay plot when a 45° polariser is used together with detector $D_1$ is 75%. The visibility is limited by the limited pulse duration and finite filters bandwidth.

Recently, a research group in Hefei, China\(^1\) has reported the generation of ten-photon polarisation entanglement [73], essentially a ten-photon GHZ states with high brightness, collection efficiency and high fidelity (91 ± 1%) spontaneous parametric down-conversion (SPDC) source. They used a similar architecture as in Figure 23 but instead of just pumping the photons into the beta-barium borate (BBO) crystal to generate photon pairs with relatively low pair efficiency (probability for generating a single pair is about 0.1 to restrict the noise due to the generation of more than one photon pairs), a setup with a half-wave plate sandwiched between the two BBO crystals creates a polarisation-entangled photons with high two-photon count rates per second (about few millions). They managed to employ five of the similar setups at the same time to produce ten-photon polarisation-entangled GHZ state with fidelity of 0.573 ± 0.023 and count rate of about 0.001 Hz, which is three orders lower than the eight-photon GHZ state ($\approx$ 0.2 Hz).

However, despite the fact that phase super-sensitivity has been achieved in systems using either the GHZ or the N00N state, they are known for their difficulty in preparing high purity states and the sensitivity to losses which hinder the scalability of the quantum state to higher photon numbers. In 1993, Holland and Burnett [40] from the University of Oxford introduced an alternative scheme for generating scalable entangled states that can be demonstrated using the current hardware, e.g. polarising beam splitter and Mach-Zehnder without any post-selection at the detection stage. The scheme involves injecting twin Fock states into the two inputs of the Mach-Zehnder. The twin Fock input state $|\frac{N}{2}, \frac{N}{2}\rangle$ impinges on the first beam splitter. A relative phase shift $\phi$ is induced on either arm of the interferometer. The output state is known as the Holland-Burnett state, which is defined as

$$|\psi\rangle = \sum_{n=0}^{N/2} C_n |2n, N-2n\rangle, \quad C_n = \frac{\sqrt{(2n)![(N-2n)!]}}{2^{N/2}n!(N-2n-1)!} e^{2i\phi n}. \quad (4.12)$$

This quantum state approaches the same phase sensitivity as the N00N and GHZ states, which is the Heisenberg limit. The comparison between the N00N state and the Holland-Burnett state, HB($n$) in terms of phase sensitivity are listed in the Ref. [74]. They showed that for $N > 4$, the HB($n$) is better compared to the maximal entangled N00N state in achieving phase super-sensitivity in lossy medium. They managed to generate HB($n$) state up to six photons, $n = 6$. Figure 24 shows the schematic of the experiment described in Ref. [74] to generate and probe a six-photon Holland-Burnett state. A Ti:Sapphire laser is pumped into a type I beta-

\(^{10}\)The same group also published an experimental eight-photon entanglement scheme in 2012. See Ref. [72].
Figure 21: The heralding scheme used in Ref. [69] to generate high purity two-photon N00N state. A pair of parametric down-conversion (PDC) sources are simultaneously pumped creating photon pairs. One from each photon pair is detected by a trigger to herald the creation of its sibling photon. The heralded photons from each pair are then impinged at a balanced beam splitter (BS1) to form a two-photon N00N state. The N00N state then propagates through the two arms of a Mach-Zehnder interferometer which induces a phase shift of $\phi$ between the arms. The two paths recombine at BS2, in which the two outputs are directed to the two photon-counting detectors $D_A$, $D_B$ respectively. The coincidence rate from trigger 1, trigger 2, $D_A$, and $D_B$ (four-fold coincidence) is monitored to obtain the two-photon interference fringes.

Barium borate (BBO) crystal to produce two type I SPDC photon pairs at 780 nm which are coupled into two polarisation-maintaining (PM) fibres. These modes are combined into a single spatial mode on a polarising beam splitter which are then passed through an interference filter (full width half maximum of 3 nm) to reduce the bandwidth of the photons. For the six-photon HB($n$) state, three SPDC photon pairs, forming a three-photon Fock states in each of the spatial mode are injected into the interferometer, i.e. $|\psi_{in}\rangle = |3,3\rangle_{H,V}$. The input state $|\psi_{in}\rangle$ is sent to the polarising beam splitter to be combined into a single spatial mode. The phase shift $\phi$ between the right and left hand circular polarisation at the output of the polarising beam splitter is applied to the output by rotating the half-wave plate by $\phi/4$. Each output of the polarising beam splitter is then sent to an array of five single photon-counting modules. Filtering on the detection of three photons in each of the interferometer output selects out the six-photons term from the source. The single measurement fringe gives a visibility of $94 \pm 2\%$. The maximum Fisher information is found to be $20.0 \pm 0.9$ which implies that the estimated phase variance is at most three times smaller than the standard quantum limit.

The sources of imperfections of the sensing scheme of the experimental scheme using the HB($n$) state are discussed in Ref. [75]. The same group also described the possibility of implementing the HB($n$) states in integrated photonics systems, where they discussed the process of generating pure, heralded bi-photons via spontaneous four-wave mixing in a birefringent fibre. They showed that by matching the bandwidth (of a spectral filter) or the spectral mode of the heralded photon, the trade-off between the heralding efficiency and purity of the states can be minimized such that the loss in purity is minimum when the heralding efficiency increases.

Other entangled states that have been demonstrated experimentally include Dicke states for $n = 4$ [70, 71] and $n = 6$ [78, 79], entangled coherent state [80] and Schrödinger cat state (e.g. coupling of an artificial atom and cavity mode in Ref. [81] and squeezed cat state in Ref. [82, 83, 84, 85]). In short, multipartite entanglement for large number of particles $N$ is extremely useful in quantum metrology due to the fact that the phase sensitivity scales as $1/N$. To date, the largest number of particles being entangled at the same time in the optical regime is the ten-photon entangled GHZ states [76] and in the atomic regime is about 3000 atoms entangled with a single photon [86]. An atomic clock measurement with eleven-fold enhancement beyond the standard quantum limit has been achieved by Hosten et al. [54] using spin-squeezed states. These advancements in quantum metrology
Figure 22: The setup schematic for the generation of N00N states by mixing quantum light and classical light. Here, the quantum light refers to the light generated from spontaneous parametric down-conversion (SPDC) which is done by pumping the pulses from Ti:Sapphire oscillator (with 120 fs pulses at a repetition rate of 80 MHz) to a lithium triborate (LBO) crystal to achieve frequency doubling to obtain a 404 nm ultraviolet (UV) pulses. These pulses are then pumped into a beta-barium borate (BBO) crystal to undergo degenerate type I spontaneous parametric down-conversion (SPDC). The classical light refers to the attenuated pulse (using an attenuator and a liquid crystal (LC) phase retarder) from the Ti:Sapphire laser. These two lights (quantum light in $\hat{H}$ polarization and classical light in $\hat{V}$ polarization) combine at a polarising beam splitter which form a $N$-photon N00N state. The state is then verified by a Mach-Zehnder interferometer and arrays of photon counting detectors. The scheme works for any $N$-photon N00N state without much modification except in the usage of the detectors to resolve $N$-photons event and optimising the parameters for the production pair amplitude ratio of the coherent state and SPDC inputs, $\gamma = |\alpha|^2/r$, where $\alpha$ is the amplitude of the displacement operator and $r$ is the amplitude of the squeezing operator. Source: Ref. [64].

pave the way for these non-classical states to be implemented in the current time metrology standard and in the detection of gravitational waves in the near future.

5 Summary

In metrology, the measurement of a parameter need to be highly sensitive so that it can be measured up to a larger precision. The standard quantum limit used to be the absolute limit that any system can achieve, but now since the introduction of quantum mechanics, the standard quantum limit can be beaten and the Heisenberg limit replaces its role as the absolute limit of sensitivity. The techniques used and the advancement in phase metrology are discussed in this paper. These are done on interferometers, notably the Mach-Zehnder and the Ramsey interferometer which are described in Section 2. They are among the most sensitive instruments to obtain the measured parameter, for instance the phase by superimposing electromagnetic waves propagate inside the interferometer. From the interferometer, the phase estimation is usually done by extracting the variance and the slope of the interference fringes. The phase estimation sensitivity is limited by the quantum Cramér-Rao bound which is related to the quantum Fisher information. The quantum Fisher information is a strategy in optimal phase estimation that has a value equal to four times the variance of the operator associated with the phase measurement.

Various entanglement indicators and measures are used to characterise the correlation or entanglement of the non-classical states used in quantum-enhanced phase metrology. They are briefly described in Section 3. Fidelity and the visibility of the interference fringes generated by the coincidence measurements are the most commonly used indicators for the non-classicality of the quantum state. They are used to characterise the entanglement depth of the generated non-classical states in experiments described in Section 4.

Section 4.1 describes the properties and the methods to generate squeezed states in optical and atomic systems. Squeezed states are the most commonly used for quantum phase metrology. They have a lower uncertainty in one of the quadratures in the phase space compared to the coherent state but yet it still saturates the Heisenberg uncertainty relation just like the coherent state. This can be generated by non-linear optical processes, mainly by optical parametric generation (e.g. spontaneous parametric down-conversion and four-
wave mixing). These are used to generate bi-photons, i.e. a pair of correlated photons which is essential as the starting point for generating entangled states like the NO0N, GHZ and Schrödinger cat state. These states have been shown theoretically to be able to achieve the Heisenberg limit, however due to different losses only sub-standard quantum limit precision can be achieved experimentally. The Holland-Burnett state has been shown in Ref. [40, 75, 74] to be an alternative to NO0N state. It is loss-tolerant and it can be easily scaled to large number of systems. The entanglement depth of the states generated using the experimental schemes described in Section 4.1 and Section 4.2 are summarised in Table 1. In general, the generated states will need to have a fidelity greater than 0.5, visibility greater than \( \approx 71\% \) and/or non-zero squeezing to be non-classical or entangled. However, for NO0N state, if the visibility is greater than the classical limit (33.3\% \( N = 2 \), 10\% \( N = 3 \) and 0.79\% \( N = 5 \)), then the state exhibits non-classicality [87].

Apart from the optical and atomic regimes, the mechanical regime has been shown recently to exhibit potentials in achieving phase sensitivity in a macroscopic scale. This method, introduced by the Aspelmeyer group from University of Vienna exploits the radiation pressure from pulsed optical sources on a mechanical oscillator to produce interference fringes. This method has been report recently in Ref. [88], where the mechanical motion can be made non-classical without the oscillator being coupled to a photon source. They claimed that their experimental scheme can be easily scalable for larger superposition states and is optical loss tolerant. Clark et al [89] have shown by exploiting the optomechanical interactions between a mechanical oscillator and the squeezed light, such that the radiation pressure dominates over the thermal noise, the mechanical oscillator can improve on the phase measurements on the non-classical squeezed light. This method looks promising and would benefit other ongoing projects, particularly in the detection of the gravitational waves.
Figure 24: A schematic of the interferometry involving the twin Fock state or the Holland-Burnett state. To generate the six-photon Holland-Burnett state, a type I spontaneous parametric down-conversion (SPDC) source, a beta-barium borate (BBO) crystal supplies the interferometer with three indistinguishable photons in each of the two spatial mode, forming the twin three-photon Fock state $|_{3,3}\rangle$. These modes are recombined at the polarising beam splitter into a single spatial mode. Then the photons undergo a phase shift $\phi$ between the left- and right- circular polarisation due to the presence of the half-wave plate. The second beam splitter then recombines the two circular polarised modes and the outputs are measured by the photon counting module arrays. Filtering on detections of 3 photons in each of the interferometer outputs gives the 6-photon term from the down-conversion source. Source: Ref. [74].

References


Table 1: A summary of the experiments described in Section 4.1 (Squeezed states) and 4.1.3 (Spin-squeezed states) and 4.2 (EPR, N00N, GHZ and Holland-Burnett states), arranged by the order of appearance. Here, the abbreviations CV, HOM and NV are continuous-variable, Hong-Ou-Mandel effect and nitrogen vacancy respectively. The entanglement measure for squeezing is defined as the uncertainty in one of the quadratures of the squeezed states relative to that of the vacuum state, i.e. below the standard noise limit. \( N \) is the number of entangled entities of the system, i.e. it may refer to the \( N \)-photon N00N state or the number of optically coupled trapped atoms. The visibility labeled with * is associated with the \( N = 5 \) N00N state. For the \( N < 5 \) N00N states, the associated visibilities are \( \leq 74 \% \). For the † case, the visibility listed here is associated with the \( N = 6 \) Holland-Burnett state.

<table>
<thead>
<tr>
<th>State</th>
<th>Authors</th>
<th>Platform/system</th>
<th>Fidelity</th>
<th>Entanglement Measure</th>
<th>Visibility (%)</th>
<th>Squeezing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Squeezed</td>
<td>Wu et al</td>
<td>Optical</td>
<td>-</td>
<td>-</td>
<td>50%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Preble et al</td>
<td>On-chip micro resonator</td>
<td>-</td>
<td>93.3 ± 2.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Suder et al</td>
<td>Trapped atoms</td>
<td>-</td>
<td>0.3 dB</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spin-Squeezed</td>
<td>Gross et al</td>
<td>Bose-Einstein Condensate</td>
<td>-</td>
<td>8.2 dB</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Hosten et al</td>
<td>Trapped atoms (( N \sim 10^3 ))</td>
<td>-</td>
<td>20.1 ± 0.3 dB</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Muessel et al</td>
<td>Trapped atoms (( N = 12,300 ))</td>
<td>-</td>
<td>5.3 ± 0.5 dB</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Togan et al</td>
<td>NV centre (diamond)</td>
<td>0.69 ± 0.07</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EPR (CV entangled)</td>
<td>Manaa et al</td>
<td>Photonic</td>
<td>-</td>
<td>99.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N00N</td>
<td>Mitchell et al</td>
<td>Optical-Dark ports (( N = 3 ))</td>
<td>-</td>
<td>42 ± 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Smith et al</td>
<td>Optical-Heralded scheme (( N = 2 ))</td>
<td>-</td>
<td>≥ 63</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ulanov et al</td>
<td>Optical-Reverse HOM (( N = 2 ))</td>
<td>≥ 0.88</td>
<td>-</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Aek et al</td>
<td>Optical-Mixing classical and quantum light (up to ( N = 5 ))</td>
<td>-</td>
<td>42 ± 2*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GHZ</td>
<td>Bouwmeester et al</td>
<td>Optical (( N = 3 ))</td>
<td>-</td>
<td>75</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Wang et al</td>
<td>Optical (( N = 10 ))</td>
<td>-</td>
<td>91 ± 1</td>
<td></td>
<td></td>
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<tr>
<td>Holland-Burnett</td>
<td>Xiang et al</td>
<td>Optical (up to ( N = 6 ))</td>
<td>-</td>
<td>94 ± 2†</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


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